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INVARIANTS AND K-SPECTRUMS OF LOCAL THETA LIFTS

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ABSTRACT. Let (G, G') be a type I irreducible reductive dual pair in $\mathrm{Sp}(W_{\mathbb{R}})$. We assume that (G, G') is under stable range where G is the smaller member. Let ρ be a genuine irreducible $(\mathfrak{g}, \widetilde{K})$-module. The first main result is to show that if ρ is unitarizable, then the full local theta lift $\rho' = \Theta(\rho)$ is an irreducible $(\mathfrak{g}', \widetilde{K}')$-module and thus equal to the local theta lift $\theta(\rho)$. The second main result is to show that the associated variety $\mathrm{AV}(\rho')$ and the associated cycle $\mathrm{AC}(\rho')$ of ρ' is obtained via the notion of local theta lifts of nilpotent orbits and cycles from those of ρ. Finally we obtain some interesting $(\mathfrak{g}, \widetilde{K})$-modules whose $\widetilde{K}$-spectrums are isomorphic to the spaces of global sections of vector bundles on nilpotent $K_{\mathbb{C}}$-orbits in $\mathfrak{p}^*$.

1. INTRODUCTION

Let $W_{\mathbb{R}}$ be a symplectic real vector space. Throughout this paper (G, G') will denote a Type I irreducible reductive dual pair in $\mathrm{Sp}(W_{\mathbb{R}})$. We assume that (G, G') is under stable range where G is the smaller member. Such dual pairs are listed in Table 1 in Section 2.1. Let ρ be a genuine irreducible $(\mathfrak{g}, \widetilde{K})$-module. The first main result is to show that if ρ is unitarizable, then the full local theta lift $\rho' = \Theta(\rho)$ is an irreducible $(\mathfrak{g}', \widetilde{K}')$-module and thus equal to the local theta lift $\theta(\rho)$. The second main result is to show that the associated variety $\mathrm{AV}(\rho')$ and the associated cycle $\mathrm{AC}(\rho')$ of ρ' is obtained via the notion of local theta lifts of nilpotent orbits and cycles from those of ρ. Finally we obtain some interesting $(\mathfrak{g}, \widetilde{K})$-modules whose $\widetilde{K}$-spectrums are isomorphic to the spaces of global sections of vector bundles on nilpotent $K_{\mathbb{C}}$-orbits in $\mathfrak{p}^*$.

1.1. Let $\widetilde{\mathrm{Sp}}(W_{\mathbb{R}})$ be the metaplectic double cover of $\mathrm{Sp}(W_{\mathbb{R}})$. For any subgroup E of $\mathrm{Sp}(W_{\mathbb{R}})$, let $\widetilde{E}$ denote its inverse image in $\widetilde{\mathrm{Sp}}(W_{\mathbb{R}})$. We choose a maximal compact subgroup U of $\mathrm{Sp}(W_{\mathbb{R}})$ such that $K := G \cap \mathrm{U}$ and $K' := G' \cap \mathrm{U}$ are maximal compact subgroups of G and G' respectively. The choice of U is equivalent to a complex structure on $W_{\mathbb{R}}$ with a positive definite Hermitian form $\langle , \rangle$. Then $\mathrm{U} = \mathrm{U}(W, \langle , \rangle)$ is the unitary group on the underlying complex vector space W.

We follow the notation in [12] closely. The Fock module $\mathscr{Y}$ of the oscillator representation of $\widetilde{\mathrm{Sp}}(W_{\mathbb{R}})$ is realized as the space $\mathbb{C}[W]$ of complex polynomials on W. Let ς denote the minimal $\widetilde{\mathrm{U}}$-type of $\mathscr{Y}$. It is a one dimensional character of $\widetilde{\mathrm{U}}$ acting on the space of constant functions in $\mathbb{C}[W]$. Let $\varsigma|_{\widetilde{E}}$ denote the restriction of ς to $\widetilde{E}$ for any subgroup E of U.

Let $\mathfrak{g}$ and $\widetilde{K}$ denote the complexified Lie algebra and maximal compact subgroup of $\widetilde{G}$ respectively. Let ρ be an irreducible admissible $(\mathfrak{g}, \widetilde{K})$-module. By (2.5) in [12],

$$\mathscr{Y} / (\cap_{\psi \in \mathrm{Hom}_{\mathfrak{g}, \widetilde{K}}(\mathscr{Y}, \rho)} \ker \psi) \simeq \rho \otimes \Theta(\rho) \tag{1}$$

where $\Theta(\rho)$ is a $(\mathfrak{g}', \widetilde{K}')$-module called the *full (local) theta lift* of ρ. Theorem 2.1 in [12] states that if $\Theta(\rho) \neq 0$, then $\Theta(\rho)$ is a $(\mathfrak{g}', \widetilde{K}')$-module of finite length with an infinitesimal character and it has an unique irreducible quotient $\theta(\rho)$ called the *(local) theta lift* of ρ'. Moreover $\theta(\rho_1)$ and $\theta(\rho_2)$ are isomorphic irreducible admissible $(\mathfrak{g}', \widetilde{K}')$-modules if and only if ρ_1 and ρ_2 are isomorphic irreducible admissible $(\mathfrak{g}, \widetilde{K})$-modules.

It is a result of [28] that in the stable range $\theta(\rho)$ is nonzero. This partly generalizes a previous result of [17] which states that if ρ is irreducible and unitarizable, then $\theta(\rho)$ is nonzero and unitarizable.

In order to state our first result, we need to exclude following special case.

(†) The dual pair $(G, G') = (\mathrm{Sp}(n, \mathbb{R}), \mathrm{O}(2n, 2n))$ and ρ is the one dimensional genuine representation of $\widetilde{\mathrm{Sp}}(n, \mathbb{R})$.

We state our first main result.

Theorem A. *Suppose (G, G') is in stable range where G is the smaller member. Let ρ be an irreducible unitarizable $(\mathfrak{g}, \widetilde{K})$-module. We exclude the case (†) above. Then*

$$\Theta(\rho) = \theta(\rho)$$

as $(\mathfrak{g}', \widetilde{K}')$-modules. In other words, $\Theta(\rho)$ is already irreducible and unitarizable.

The proof is given in Section 2.3.

In Case (†), $\Theta(\rho)$ is reducible by the Lee's appendix in [18].

The above theorem is useful because in applying see-saw pairs argument, one has to use $\Theta(\rho)$ instead of $\theta(\rho)$. For example, we could deduce a formula for the $\widetilde{K}'$-types of $\Theta(\rho)$ in Proposition 2.2.

1.2. Before stating other results, we briefly review the definitions of some invariants of a $(\mathfrak{g}, K)$-module. See Section 2 in [30] for details.

Let ϱ be a $(\mathfrak{g}, K)$-module of finite length and let $0 \subset F_0 \subset \cdots \subset F_j \subset F_{j+1} \subset \cdots$ be a good filtration of ϱ. Then $\mathrm{Gr}\,\varrho = \bigoplus F_j/F_{j-1}$ is a finite generated $(\mathcal{S}(\mathfrak{p}), K)$-module where $\mathcal{S}(\mathfrak{p})$ is the symmetric algebra on $\mathfrak{p}$.

Let $\mathscr{A}$ be the associated $K_{\mathbb{C}}$-equivariant coherent sheaf of $\mathrm{Gr}\,\varrho$ on $\mathfrak{p}^* = \mathrm{Spec}\,(\mathcal{S}(\mathfrak{p}))$. The *associated variety* of ϱ is defined to be $\mathrm{AV}(\varrho) := \mathrm{Supp}(\mathscr{A})$ in $\mathfrak{p}^*$. Its dimension is called the *Gelfand-Kirillov dimension* of ϱ. It is a well known fact that $\mathrm{AV}(\varrho)$ is a closed subset of the null cone of $\mathfrak{p}^*$.

Let $\mathrm{AV}(\varrho) = \bigcup_{j=1}^r \overline{\mathcal{O}_j}$ such that $\mathcal{O}_j$ are open $K_{\mathbb{C}}$-orbits in $\mathrm{AV}(\varrho)$. By Lemma 2.11 in [30], there is a finite $(\mathcal{S}(\mathfrak{p}), K_{\mathbb{C}})$-invariant filtration $0 \subset \mathscr{A}_0 \subset \cdots \subset \mathscr{A}_l \subset \cdots \subset \mathscr{A}_n = \mathscr{A}$ of $\mathscr{A}$ such that $\mathscr{A}_l/\mathscr{A}_{l-1}$ is generically reduced on each $\overline{\mathcal{O}_j}$. For a closed point $x_j \in \mathcal{O}_j$, let $i_{x_j} \colon \{x_j\} \hookrightarrow \mathcal{O}_j$ be the natural inclusion map and let K_{x_j} be the stabilizer of x_j in $K_{\mathbb{C}}$. Now

$$\chi_{x_j} := \bigoplus\nolimits_l (i_{x_j})^*(\mathscr{A}_l/\mathscr{A}_{l-1})$$

is a finite dimensional rational representation of K_{x_j}. We call χ_{x_j} an *isotropy representation* of ϱ at x_j. Its image $[\chi_{x_j}]$ in the Grothendieck group of finite dimensional rational K_x-modules is called the *isotropy character* of ϱ at x_j. The the isotropy representation is dependent on the filtration but the isotropy character is independent of the filtration.

We call $\{\, (\mathcal{O}_j, x_j, \chi_{x_j}) : j = 1, \ldots, r \,\}$ the set of *orbit data attached to the filtration $\mathscr{A}_j$*. On the other hand, $\{\, (\mathcal{O}_j, x_j, [\chi_{x_j}]) : j = 1, \ldots, r \,\}$ is independent of the filtration and we call it the set of *orbit data attached to* ϱ. Two orbit data are equivalent if they are conjugate to each other by the $K_{\mathbb{C}}$-action. We define the *multiplicity of ϱ at $\mathcal{O}_j$* to be $m(\mathcal{O}_j, \varrho) = \dim_{\mathbb{C}} \chi_{x_j}$ and the *associated cycle* of ϱ to be $\mathrm{AC}(\varrho) = \sum_{j=1}^r m(\mathcal{O}_j, \varrho)[\overline{\mathcal{O}_j}]$.

In summary the associated variety, the associated cycle and isotropy character(s) are invariants of ϱ, i.e. they are independent of the choices of filtrations.

We add that the above invariants of ϱ and of its contragredient ϱ^* are related by a Chevalley involution C on G. See [1] or Appendix B.

1.3. Now we describe a result about the associated variety of $\Theta(\rho)$.

Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and $\mathfrak{g}' = \mathfrak{k}' \oplus \mathfrak{p}'$ denote the complexified Cartan decompositions of $\mathfrak{g}$ and $\mathfrak{g}'$ respectively. There are two moment maps

$$\mathfrak{p}^* \xleftarrow{\phi} W \xrightarrow{\phi'} \mathfrak{p}'^*. \tag{2}$$

The maps ϕ and ϕ' are given explicitly in Table 2. For a $K_{\mathbb{C}}$-invariant closed subset S of $\mathfrak{p}^*$, we define *the theta lift of S* to be $\theta(S) = \phi'(\phi^{-1}S)$, which is a $K'_{\mathbb{C}}$-invariant closed subset of $\mathfrak{p}'^*$ (see Appendix A.1). Let

$$N(\mathfrak{p}^*) = \left\{\, x \in \mathfrak{p}^* \;\middle|\; 0 \in \overline{K_{\mathbb{C}} \cdot x} \,\right\} \tag{3}$$

be the null cone of $\mathfrak{p}^*$ under the $K_{\mathbb{C}}$-action. Let $\mathfrak{N}_{K_{\mathbb{C}}}(\mathfrak{p}^*)$ be the set of nilpotent $K_{\mathbb{C}}$-orbits in $\mathfrak{p}^*$. We define $N(\mathfrak{p}'^*)$ and $\mathfrak{N}_{K'_{\mathbb{C}}}(\mathfrak{p}'^*)$ in the same way. By Theorem A.2, $\theta(S) \subseteq N(\mathfrak{p}'^*)$ if $S \subseteq N(\mathfrak{p}^*)$.

Since $\Theta(\rho)$ is of finite length, the associated variety $\mathrm{AV}(\Theta(\rho))$ of $\Theta(\rho)$ is a closed subvariety of $N(\mathfrak{p}'^*)$.

Theorem B. *For any real reductive dual pair (not necessary of type I or in stable range). We have an upper bound of the associated variety of $\Theta(\rho)$:*

$$\mathrm{AV}(\Theta(\rho)) \subseteq \theta(\mathrm{AV}(\rho^*))).$$

The proof is given in Section 3.6.

The above theorem is a correction to Proposition 3.12 in Nishiyama-Zhu [25].

1.4. We assume (G, G') is in the stable range where G is the smaller member. Given $\mathcal{O} \in \mathfrak{N}_{K_{\mathbb{C}}}(\mathfrak{p}^*)$, it is a result of [4] and [24] that there is a unique $\mathcal{O}' \in \mathfrak{N}_{K'_{\mathbb{C}}}(\mathfrak{p}'^*)$ such that $\overline{\mathcal{O}'} = \theta(\overline{\mathcal{O}})$. We say that $\mathcal{O}'$ is the *theta lift of $\mathcal{O}$* and we denote $\mathcal{O}' = \theta(\mathcal{O})$. Moreover,

$$\begin{aligned} \theta\colon \mathfrak{N}_{K_{\mathbb{C}}}(\mathfrak{p}^*) &\to \mathfrak{N}_{K'_{\mathbb{C}}}(\mathfrak{p}'^*) \\ \mathcal{O} &\mapsto \mathcal{O}' \end{aligned}$$

is an injective map preserving the order on both sides given by closure relations, i.e. $\theta(\mathcal{O}_2) \subset \overline{\theta(\mathcal{O}_1)}$ if $\mathcal{O}_2 \subset \overline{\mathcal{O}_1}$. See Theorem A.2.

Definition 1.5. *We define following notion of theta lifts of objects in stable range.*

(1) Let $c = \sum_j m_j[\overline{\mathcal{O}_j}]$ be a formal sum of nilpotent orbits. We define the theta lift of the cycle c to be $\theta(c) := \sum_j m_j[\overline{\theta(\mathcal{O}_j)}]$.

(2) Let $(\mathcal{O}, x, \chi_x)$ be an orbit datum where $\mathcal{O} \in \mathfrak{N}_{K_{\mathbb{C}}}(\mathfrak{p}^*)$, $x \in \mathcal{O}$ and χ_x is a finite-dimensional rational $\widetilde{K}_x$-module. Let $\mathcal{O}' = \theta(\mathcal{O})$. Fixing points $x \in \mathcal{O}$, $w \in W$, $x' \in \mathcal{O}'$ such that $\phi(w) = x$ and $\phi'(w) = x'$, we define a group homomorphism $\alpha\colon K'_{x'} \to K_x$ in (35). We define the theta lift of the orbit datum $(\mathcal{O}, x, \chi_x)$ to be $(\mathcal{O}', x', \chi_{x'})$ where

$$\chi_{x'} := \varsigma|_{\widetilde{K}'_{x'}} \otimes (\varsigma|_{\widetilde{K}_x} \otimes \chi_x) \circ \alpha.$$

We denote $\theta(\mathcal{O}, x, \chi_x) = (\mathcal{O}', x', \chi_{x'})$. Similarly we define the theta lift of $(\mathcal{O}, x, [\chi_x])$ to be $(\mathcal{O}', x', [\chi_{x'}])$. We denote $\theta(\mathcal{O}, x, [\chi_x]) = (\mathcal{O}', x', [\chi_{x'}])$.

Theorem C. *Suppose (G, G') is in stable range where G is the smaller member. Let ρ be a genuine irreducible $(\mathfrak{g}, \widetilde{K})$-module. Suppose $\{\, (\mathcal{O}_j, x_j, [\chi_{x_j}]) : j = 1, \ldots, r \,\}$ is the set of orbit data attached to ρ^*. Then $\Theta(\rho)$ is attached to the set of orbit data $\{\, \theta(\mathcal{O}_j, x_j, [\chi_{x_j}]) : j = 1, \ldots, r \,\}$.*

The next theorem is a corollary of Theorem C.

Theorem D. *Suppose (G, G') is in stable range where G is the smaller member. Then*

$$\mathrm{AV}(\Theta(\rho)) = \theta(\mathrm{AV}(\rho^*)) \text{ and } \mathrm{AC}(\Theta(\rho)) = \theta(\mathrm{AC}(\rho^*)).$$

In particular if ρ is unitarizable, then $\mathrm{AV}(\theta(\rho)) = \theta(\mathrm{AV}(\rho^*))$ and $\mathrm{AC}(\theta(\rho)) = \theta(\mathrm{AC}(\rho^*))$ by Theorem A.

The proofs of Theorems C and D are given in Section 4.6. In these two Theorems, we do not require that ρ^* is unitarizable. We will show in the proof of Lemma 4.7 that every $\theta(\overline{\mathcal{O}_j})$ has the same dimension equal to $\dim \mathrm{AV}(\Theta(\rho))$, i.e. the Gelfand-Kirillov dimension of $\Theta(\rho)$. However there are examples where $\Theta(\rho)$ is reducible and $\theta(\rho)$ has smaller Gelfand-Kirillov dimension than that of $\Theta(\rho)$. In particular $\mathrm{AV}(\theta(\rho))$ does not contain any $\theta(\mathcal{O}_j)$.

Theorem D overlaps with the previous work of [23] and [32] where G is a compact group. It also extends the work [25] where ρ is a unitarizable lowest weight modules.

For a $(\mathfrak{g}, K)$-module ϱ of finite length, we define $\mathrm{V}_{\mathbb{C}}(\varrho)$ to be the complex variety cut out by the ideal $\mathrm{Gr}\,(\mathrm{Ann}_{\mathcal{U}(\mathfrak{g})}\varrho)$ in $\mathfrak{g}^*$, where $\mathrm{Gr}\,(\mathrm{Ann}_{\mathcal{U}(\mathfrak{g})}\varrho)$ is the graded ideal of $\mathrm{Ann}_{\mathcal{U}(\mathfrak{g})}\rho$ in $\mathrm{Gr}\,\mathcal{U}(\mathfrak{g}) = \mathbb{C}[\mathfrak{g}^*]$. Alternatively $\mathrm{V}_{\mathbb{C}}(\varrho)$ is the associated variety of the $(\mathfrak{g}\oplus\mathfrak{g}, G_{\mathbb{C}})$-module $\mathcal{U}(\mathfrak{g})/\mathrm{Ann}_{\mathcal{U}(\mathfrak{g})}\varrho$. It is an $\mathrm{Ad}^*G_{\mathbb{C}}$-invariant complex variety in $\mathfrak{g}^*$ whose dimension is equal to $2\dim \mathrm{AV}(\varrho)$. By Proposition B.1, $\mathrm{V}_{\mathbb{C}}(\varrho^*) = \mathrm{V}_{\mathbb{C}}(\varrho)$.

We recall that (G, G') is a type I irreducible dual pair in stable range where G is the smaller member. Correspondingly we have two moment maps

$$\mathfrak{g}^* \xleftarrow{\phi_G} W_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} \xrightarrow{\phi_{G'}} \mathfrak{g}'^*. \tag{4}$$

For an $\mathrm{Ad}^*G_{\mathbb{C}}$-invariant complex subvariety S of $\mathfrak{g}^*$, we define $\theta_{\mathbb{C}}(S) = \phi_{G'}\phi_G^{-1}(S)$. This is an $\mathrm{Ad}^*G'_{\mathbb{C}}$-invariant complex subvariety of $\mathfrak{g}'^*$. We state a corollary of Theorem D.

Corollary E. *Suppose (G, G') is in stable range where G is the smaller member. Then*

$$\mathrm{V}_{\mathbb{C}}(\Theta(\rho)) = \theta_{\mathbb{C}}(\mathrm{V}_{\mathbb{C}}(\rho)).$$

The proof is given in Section 4.8.

The above corollary overlaps with Theorem 0.9 in [29] where Prezbinda proves the identity $\mathrm{V}_{\mathbb{C}}(\theta(\rho)) = \theta_{\mathbb{C}}(\mathrm{V}_{\mathbb{C}}(\rho))$ for dual pairs and unitarizable ρ satisfying some technical conditions.

We would like to relate a recent result of [8] where Gomez and Zhu show that the dimensions of the generalized Whittaker functionals of the Casselman-Wallach globalizations of ρ and $\Theta(\rho)$ are the same. It is a famous result of [21] that in the p-adic case, the dimensions of the spaces of generalized Whittaker functionals of an algebraic irreducible representation are equal to the corresponding multiplicities in its wavefront cycle. Theorem D together with [8] could be interpreted as an evidence for the corresponding result for real classical groups.

1.6. In Section 5, we consider representations whose K-spectrums are the same as the global sections of $K_{\mathbb{C}}$-equivariant algebraic vector bundles on nilpotent orbits. We will show that theta lifts in stable range preserve such property.

Let $\mathcal{O} \in \mathfrak{N}_{K_{\mathbb{C}}}(\mathfrak{p}^*)$ and $x \in \mathcal{O}$. Let $\pi : K_{\mathbb{C}} \to \mathcal{O}$ be the natural projection map given by $\pi(k) = (\mathrm{Ad}^*k)x$. For a finite dimensional rational K_x-module (χ_x, V_x), we define a $K_{\mathbb{C}}$-equivariant pre-sheaf on $\mathcal{O}$ by $\mathscr{L}(U) = (\mathbb{C}[\pi^{-1}(U)] \otimes_{\mathbb{C}} V_x)^{K_x}$ for all open subsets U of $\mathcal{O}$. By [3], $\mathscr{L}$ is a sheaf. We define a $(\mathbb{C}[\mathcal{O}], K_{\mathbb{C}})$-module

$$\mathrm{Ind}_{K_x}^{K_{\mathbb{C}}}\chi_x = (\mathbb{C}[K] \otimes V_x)^{K_x} = H^0(\mathcal{O}, \mathscr{L}).$$

If $(\mathcal{O}, x, \chi_x)$ appears in the orbit data attached to a filtration of a finite length $(\mathfrak{g}, K)$-module, then we have

$$\mathscr{L} \text{ is generated by its space of global sections} \mathrm{Ind}_{K_x}^{K_{\mathbb{C}}} V_x. \tag{5}$$

For the rest of the paper we will always assume that data $(\mathcal{O}, x, \chi_x)$ satisfy (5).

We exclude following special cases:

$$\begin{aligned} &(G, G') = (\mathrm{Sp}(2n, \mathbb{R}), \mathrm{O}(p, q)) \text{ where } p = 2n \text{ or } q = 2n; \\ &(G, G') = (\mathrm{Sp}(2n, \mathbb{C}), \mathrm{O}(4n, \mathbb{C})). \end{aligned} \tag{††}$$

Theorem F. *Suppose (G, G') is in the stable range with G the smaller member excluding (††). Let ρ be an irreducible admissible genuine $(\mathfrak{g}, \widetilde{K})$-module. Let $(\mathcal{O}, x, \chi_x)$ be an orbit datum satisfying (5) such that, as $\widetilde{K}_{\mathbb{C}}$-modules*

$$\rho^* = \mathrm{Ind}_{\widetilde{K}_x}^{\widetilde{K}_{\mathbb{C}}} \chi_x.$$

Let $(\mathcal{O}', x', \chi_{x'})$ be the theta lifting of $(\mathcal{O}, x, \chi_x)$. Then, as $\widetilde{K}'$-modules,

$$\Theta(\rho) = \mathrm{Ind}_{\widetilde{K}'_{x'}}^{\widetilde{K}'_{\mathbb{C}}} \chi_{x'}. \tag{6}$$

The above theorem is a corollary of Proposition 5.1 which is a statement on the level of $(\mathcal{S}(\mathfrak{p}), \widetilde{K}_{\mathbb{C}})$-modules.

1.7. We relate our results with a conjecture of Vogan on geometric quantizations and unipotent representations.

Definition 1.8 (Definition 7.13 in [30]). *Let $\mathcal{O} \in \mathfrak{N}_{K_{\mathbb{C}}}(\mathfrak{p}^*)$ and $x \in \mathcal{O}$. The stabilizer K_x acts on the cotangent space $\mathrm{T}_x^*\mathcal{O} \cong (\mathfrak{k}/\mathfrak{k}_x)^*$. We define a character of K_x by*

$$\gamma_x(k) = \det(\mathrm{Ad}(k)|_{(\mathfrak{k}/\mathfrak{k}_x)^*}) \qquad k \in K_x. \tag{7}$$

An algebraic representation of the double cover $\widetilde{K}_x$ is called admissible if

$$\chi_x(\exp(X)) = \gamma_x(\exp(X/2)) \cdot \mathrm{Id} \qquad \forall X \in \mathfrak{k}_x. \tag{8}$$

An orbit datum $(\mathcal{O}, x, [\chi_x])$ is called an admissible orbit datum if χ_x is an admissible representation of K_x. An orbit $\mathcal{O} \in \mathfrak{N}_{K_{\mathbb{C}}}(\mathfrak{p}^*)$ is called admissible if it is part of an admissible data. A representation χ_x of K_x satisfying (8) is uniquely determined by its character $[\chi_x]$.

A $(\mathfrak{g}, \widetilde{K})$-module ρ is said to have $\widetilde{K}$-spectrum determined by an admissible orbit datum $(\mathcal{O}, x, [\chi_x])$ if

$$\rho|_{\widetilde{K}_{\mathbb{C}}} = \mathrm{Ind}_{\widetilde{K}_x}^{\widetilde{K}_{\mathbb{C}}} \chi_x \tag{9}$$

as a $\widetilde{K}_{\mathbb{C}}$-module. Such a representation ρ could be considered as a quantization of the orbit $\mathcal{O}$. In Conjecture 12.1 in [30], Vogan conjectured that, for every admissible orbit datum $(\mathcal{O}, x, [\chi_x])$ satisfying certain technical conditions and $\partial\mathcal{O}$ has codimension at least 2 in $\overline{\mathcal{O}}$, then one can attach a unipotent representation ρ to this orbit datum and satisfies (9).

In Section 6, we will show that the notion of admissibility is compatible with theta lifts in stable range.

Proposition G. *Suppose (G, G') is in stable range where G is the smaller member. Let $(\mathcal{O}, x, [\chi_x])$ be an admissible orbit datum for group $\widetilde{G}$, Then its theta lift $\theta(\mathcal{O}, x, [\chi_x])$ is also an admissible orbit datum for the group $\widetilde{G}'$.*

The above is a consequence of Proposition 6.1.

Suppose (G, G') is in stable range where G is the smaller member and excluding (††). Let ρ be an irreducible unitarizable $(\mathfrak{g}, \widetilde{K})$-module whose $\widetilde{K}$-spectrum is given by some admissible orbit datum $(\mathcal{O}, x, [\chi_x])$. It follows from Appendix B.2 that ρ^* is an irreducible unitarizable $(\mathfrak{g}, \widetilde{K})$-module whose $\widetilde{K}$-spectrum is given by the admissible orbit datum

$$C(\mathcal{O}, x, [\chi_x]) := (C(\mathcal{O}), \mathrm{Ad}^* C(x), [\chi_x \circ C])$$

where C is a Chevalley involution on $\widetilde{G}$. By Theorem A, Theorem F and Proposition G, $\theta(\rho)$ is an irreducible unitarizable $(\mathfrak{g}', \widetilde{K}')$-module whose $\widetilde{K}'$-spectrum is given by the admissible orbit datum $\theta(C(\mathcal{O}, x, [\chi_x]))$.

1.9. Finally we construct a series of candidates for unipotent representations. Let

$$G_0, G_1, G_2, \cdots, G_n, \cdots$$

be a sequence of real classical groups such that each pair (G_n, G_{n+1}) is an irreducible type I reductive dual pair with the smaller member G_n excluding (††). Let ρ_0 be an irreducible genuine one dimensional representation of $(\mathfrak{g}_0, \widetilde{K}_0)$ such that $\rho_0|_{\mathfrak{g}_0}$ is trivial. By changing G_1 if necessary, such character always exists. It is clear that ρ_0 is attached to the admissible data $(0, 0, \rho_0|_{(\widetilde{K}_0)_\mathbb{C}})$.

Let C_n be a Chevalley involution on $\widetilde{G}_n$. Let $\rho_{n+1} = \theta(\rho_n)$ and $(\mathcal{O}_{n+1}, x_{n+1}, \chi_{n+1}) = \theta(C_n(\mathcal{O}_n, x_n, \chi_n))$. The following theorem follows from Theorem D, Theorem F and Proposition G.

Theorem H. *The $(\mathfrak{g}_n, \widetilde{K}_n)$-module ρ_n is an irreducible and unitarizable representation attached to the admissible orbit datum $(\mathcal{O}_n, x_n, \chi_n)$. Moreover, as $\widetilde{K}_n$-module,*

$$\rho_n = \mathrm{Ind}_{\widetilde{K}_{x_n}}^{(\widetilde{K}_n)_\mathbb{C}} \chi_n.$$

The above theorem generalizes a result of Yang [33] where he proved the above theorem for ρ_1. A similar result on Dimixer algebras is given in [2].

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2. The K-spectrum of unitary representation in stable range

2.1. Let (G, G') be a type I irreducible reductive dual pair in $\mathrm{Sp}(W_\mathbb{R})$. We list them in Table 1 below. We say it is in stable range with G the smaller member if it satisfies the last column in the table.

We follow the notation in [12]. By Fact 1 in [12], K' is a member of a reductive dual pair (K', M) in $\mathrm{Sp}(W_\mathbb{R})$. We form the following see-saw pair in $\mathrm{Sp}(W_\mathbb{R})$:

(10)
$$\begin{tikzcd} G' \arrow[d, dash] \arrow[dr, dash] & M \arrow[d, dash] \arrow[dl, dash] \\ K' & G \end{tikzcd}$$

The complex Lie algebra of M has Cartan decomposition $\mathfrak{m} = \mathfrak{m}^{(2,0)} \oplus \mathfrak{m}^{(1,1)} \oplus \mathfrak{m}^{(0,2)}$ where $\mathfrak{m}^{(1,1)}$ is the complexified Lie algebra of a maximal compact subgroup $M^{(1,1)}$ of M.

	G	G'	Stable range
Case $\mathbb{R}$	$\mathrm{Sp}(2n,\mathbb{R})$	$\mathrm{O}(p,q)$	$2n \leq p, q$
	$\mathrm{O}(p,q)$	$\mathrm{Sp}(2n,\mathbb{R})$	$p+q \leq n$
Case $\mathbb{C}$	$\mathrm{U}(n_1,n_2)$	$\mathrm{U}(p,q)$	$n_1+n_2 \leq p, q$
Case $\mathbb{H}$	$\mathrm{O}^*(2n)$	$\mathrm{Sp}(p,q)$	$n \leq p, q$
	$\mathrm{Sp}(p,q)$	$\mathrm{O}^*(2n)$	$2(p+q) \leq n$
Complex groups	$\mathrm{Sp}(2n,\mathbb{C})$	$\mathrm{O}(p,\mathbb{C})$	$4n \leq p$
	$\mathrm{O}(p,\mathbb{C})$	$\mathrm{Sp}(2n,\mathbb{C})$	$p \leq n$

TABLE 1. Stable range for Type I dual pairs

Let $\widetilde{\mathcal{H}}$ be the space of $\widetilde{K}'$-harmonics in $\mathscr{Y} \simeq \mathbb{C}[W]$. As an $\widetilde{M}^{(1,1)} \times \widetilde{K}'$-module,

$$\widetilde{\mathcal{H}} = \bigoplus\nolimits_{\sigma' \in \widehat{\widetilde{K}'}} \sigma \otimes \sigma' \tag{11}$$

where each σ is either zero or an irreducible genuine $\widetilde{M}^{(1,1)}$-module uniquely determined by σ'.

Proposition 2.2. *Suppose (G, G') is in stable range where G is the smaller member. Let (ρ, V_ρ) be an irreducible genuine $(\mathfrak{g}, \widetilde{K})$-module. Then*

$$\Theta(\rho)|_{\widetilde{K}'} = \bigoplus_{\sigma' \in \widehat{\widetilde{K}'}} m_{\sigma'} \sigma' = (\widetilde{\mathcal{H}} \otimes \rho^*|_{\widetilde{K}})^K$$

where $m_{\sigma'} = \dim \mathrm{Hom}_{\widetilde{K}}(\sigma, \rho)$.

If ρ is the Harish-Chandra module of a discrete series representation of $\widetilde{G}$, the above proposition is Corollary 5.3 in [11].

Proof. Let $L(\sigma)$ denote the (full) theta lift of σ', which is a unitarizable lowest weight module of $\widetilde{M}$. The pair (G, G') is in stable range implies that

$$L(\sigma) = \mathcal{U}(\mathfrak{m}) \otimes_{\mathcal{U}(\mathfrak{m}^{(1,1)} \oplus \mathfrak{m}^{(0,2)})} \sigma \cong \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{k})} \sigma \tag{12}$$

as a $\mathfrak{g}$-module. The second equality follows from Eq. (3.5) in [12]. Applying this to the see-saw pair (10), we get

$$m_{\sigma'} = \dim \mathrm{Hom}_{\widetilde{K}'}(\sigma', \rho') = \dim \mathrm{Hom}_{\mathfrak{g}, \widetilde{K}}(L(\sigma), \rho) = \dim \mathrm{Hom}_{\widetilde{K}}(\sigma, \rho). \tag{13}$$

This proves the proposition. $\square$

2.3. Let (ρ, V_ρ) be an irreducible unitarizable Harish-Chandra module of $\widetilde{G}$. For the rest of this section we will prove Theorem A.

First we recall Li's construction of $\theta(\rho)$ [17]. Define

$$\langle v_1 \otimes w_1, v_2 \otimes w_2 \rangle^{\omega}_{\rho^*} = \int_G \langle \rho^*(g) v_1, v_2 \rangle_{\rho^*} \langle \omega(g) w_1, w_2 \rangle_{\omega} \, dg$$

for all $v_1 \otimes w_1, v_2 \otimes w_2 \in V_{\rho^*} \otimes \mathscr{Y}$. All pairings are done in the completion of the Harish-Chandra modules. We set

$$R = \mathrm{Rad}(\langle , \rangle^{\omega}_{\rho^*}) = \left\{ \Phi \in V_{\rho^*} \otimes \mathscr{Y} \;\middle|\; \langle \Phi, \Phi' \rangle^{\omega}_{\rho^*} = 0, \forall \Phi' \in V_{\rho^*} \otimes \mathscr{Y} \right\}.$$

Let

$$H = (V_{\rho^*} \otimes \mathscr{Y}) / \mathrm{Rad}(\langle , \rangle^{\omega}_{\rho^*}).$$

We claim that $H \simeq \theta(\rho)$ as irreducible unitarizable Harish-Chandra modules of G'. Indeed Li [17] uses smooth vectors in the definition of $\langle , \rangle^{\omega}_{\rho^*}$ and likewise defines $H^\infty = (V_{\rho^*})^\infty \otimes$

$\mathscr{Y}^\infty/\mathrm{Rad}\langle,\rangle_{\rho^*}^\infty$. It is shown that $H^\infty = \theta(\rho)^\infty$. Clearly $H \subset H^\infty$. Since H is $\widetilde{K}'$-finite and dense in H^∞, it is the Harish-Chandra module of H^∞. This proves our claim.

We refer to (σ, V_σ) in (11) and $L(\sigma)$ in (12). Then $L(\sigma)$ is an irreducible unitarizable Harish-Chandra module of $\widetilde{M}$ and $\mathscr{Y} = \bigoplus_{\sigma' \in \widehat{\widetilde{K}'}} L(\sigma) \otimes \sigma'$.

Taking the σ'-isotropy component, we define

$$\langle v_1 \otimes w_1, v_2 \otimes w_2 \rangle_{\rho^*}^{\sigma'} = \int_{\widetilde{G}} \langle \rho^*(g) v_1, v_2 \rangle_{\rho^*} \langle g \cdot w_1, w_2 \rangle_{L(\sigma)} \, dg \quad \forall v_i \otimes w_i \in V_{\rho^*} \otimes L(\sigma).$$

We also define

$$R' = \mathrm{Rad}(\langle,\rangle_{\rho^*}^{\sigma'}) \;\text{ and }\; H(\sigma') = (V_{\rho^*} \otimes L(\sigma))\,/R'.$$

In this way

$$H = \bigoplus\nolimits_{\sigma \in \hat{K}'} H(\sigma') \otimes \sigma'.$$

We consider following embeddings:

(14)

$$H(\sigma') = (V_{\rho^*} \otimes L(\sigma))/R' \xhookrightarrow{\iota} \mathrm{Hom}_G((V_{\rho^*})^\infty \otimes L(\sigma)^\infty, \mathbb{C}) \xhookrightarrow{\text{rest}} \mathrm{Hom}_{\mathfrak{g},K}(V_{\rho^*} \otimes L(\sigma), \mathbb{C})$$

where $\iota(\Phi)$ is given by

$$\Phi \mapsto (\Phi' \mapsto \langle \Phi', \Phi \rangle_{\rho^*}^{\sigma'}) \quad \forall \Phi' \in V_{\rho^*} \otimes L(\sigma). \tag{15}$$

The last term on the right hand side of (14) is

$$\begin{aligned}
&\mathrm{Hom}_{\mathfrak{g},K}(V_{\rho^*} \otimes L(\sigma), \mathbb{C}) = \mathrm{Hom}_{\mathfrak{g},\widetilde{K}}(L(\sigma), \mathrm{Hom}_{\mathbb{C}}(V_{\rho^*}, \mathbb{C})) \\
&= \mathrm{Hom}_{\mathfrak{g},\widetilde{K}}(L(\sigma), V_\rho) \\
&\quad (L(\sigma) \text{ is } \widetilde{K}\text{-finite, so its image in } \mathrm{Hom}_{\mathbb{C}}(V_{\rho^*}, \mathbb{C}) \text{ is } \widetilde{K}\text{-finite i.e. in } V_\rho) \\
&= \mathrm{Hom}_{\widetilde{K}}(V_\sigma, V_\rho) \qquad (\text{by } (12)) \\
&= \mathrm{Hom}_{\mathbb{C}}((V_{\rho^*} \otimes V_\sigma)^K, \mathbb{C}).
\end{aligned}$$

The isomorphism between the first term and the last term in above equalities is given by restriction. By (13)

$$\dim H(\sigma') \leq \dim \mathrm{Hom}_{\widetilde{K}}(V_\sigma, V_\rho) = \dim \mathrm{Hom}_{\widetilde{K}'}(\sigma', \Theta(\rho))$$

and it is finite. We will show that (14) is an isomorphism and this will prove that $\Theta(\rho) \cong H$.

We write $V_\sigma = \bigoplus_{l \in L} D_l$ and $V_{\rho^*} = \bigoplus_{j \in J} D_j$ as direct sums of irreducible $\widetilde{K}$-modules. Then

$$\mathrm{Hom}_K(V_{\rho^*} \otimes V_\sigma, \mathbb{C}) = \bigoplus_{j \in J} \bigoplus_{l \in L} \mathrm{Hom}_K(D_j \otimes D_l, \mathbb{C}) = \bigoplus_{D_l \cong D_j^*} \mathrm{Hom}_{\mathbb{C}}((D_j \otimes D_l)^K, \mathbb{C}).$$

Let $\{d_\lambda\}$ be an orthonormal basis of D_l and let $\{d_\lambda^*\}$ be a dual basis of $D_j \simeq D_l^*$. Then $(D_j \otimes D_l)^K$ is spanned by $\mathbf{d} = \sum_\lambda d_\lambda^* \otimes d_\lambda$. It suffices to show the pairing between $V_{\rho^*} \otimes L(\sigma)$ and $\mathbf{d}$ in (15) is non-vanishing.

Globalization of $L(\sigma)$. We exhibit a globalization of the Harish-Chandra module $L(\sigma)$. Our references are [15] and [14]. We refer to M in (10). Let

$$\mathrm{Hol}(\widetilde{M}, \widetilde{M}^{(1,1)}, V_\sigma) = \left\{ f : \widetilde{M} \to V_\sigma \;\middle|\; \begin{array}{l} f \text{ is analytic,} \\ f(gk) = k^{-1} f(g) \;\; \forall g \in \widetilde{M}, k \in \widetilde{M}^{(1,1)}, \\ r(X) f = 0 \;\; \forall X \in \mathfrak{m}^{(0,2)}. \end{array} \right\}.$$

Here $r(X)$ denote the right derivative action. Let $\{ v_i \}$ be an orthonormal basis of $V_\sigma \subset L(\sigma)$. Then

$$v \mapsto \left(g \mapsto \sum_i \left\langle g^{-1}v, v_i \right\rangle_{L(\sigma)} v_i \right) \tag{16}$$

defines an injective $(\mathfrak{m}, \widetilde{M}^{(1,1)})$-module homomorphism $\xi\colon L(\sigma) \to \mathrm{Hol}(\widetilde{M}, \widetilde{M}^{(1,1)}; V_\sigma)$.

For any $g \in M$, there are unique $z(g) \in \mathfrak{m}^{(0,2)}$, $k(g) \in M_{\mathbb{C}}^{(1,1)}$, $z'(g) \in \mathfrak{m}^{(2,0)}$ such that $g = \exp(z(g))k(g)\exp(z'(g))$. Let Ω denote the image of z. Let $\zeta : \widetilde{M} \to M \xrightarrow{z} \Omega$ be the composite map. Then $\Omega = \{z(g) \in \mathfrak{m}^{(0,2)} : g \in M\} \simeq M/M^{(1,1)} = \widetilde{M}/\widetilde{M}^{(1,1)}$ is a bounded open subset in $\mathfrak{m}^{(0,2)}$ and

$$M \subset \exp(\Omega) \cdot M_{\mathbb{C}}^{(1,1)} \cdot \exp(\mathfrak{m}^{(2,0)}).$$

Let $\mathrm{Hol}(\Omega, V_\sigma)$ denote the space of holomorphic functions on Ω with values in V_σ. We define $\mathrm{P} : \mathrm{Hol}(\widetilde{M}, \widetilde{M}^{(1,1)}, V_\sigma) \to \mathrm{Hol}(\Omega, V_\sigma)$ in the following way: For $f \in \mathrm{Hol}(\widetilde{M}, \widetilde{M}^{(1,1)}, V_\sigma)$, we set $\mathrm{P}f \in \mathrm{Hol}(\Omega, V_\sigma)$ by $\mathrm{P}f(g\widetilde{M}^{(1,1)}) = k(g)f(g)$. Then P is a bijection.

Let $\overline{\xi} = \mathrm{P} \circ \xi : L(\sigma) \to \mathrm{Hol}(\Omega, V_\sigma)$. The space of lowest $\widetilde{M}^{(1,1)}$-type V_σ in the image of $\overline{\xi}$ is the space of constant functions on Ω. Let $\mathbb{C}[\mathfrak{m}^{(0,2)}]$ denote the space of polynomials on $\mathfrak{m}^{(0,2)}$. Then $\overline{\xi}(L(\sigma))$ is the linear span of

$$\left\{ p \times \overline{\xi}(v) \;\middle|\; p \in \mathbb{C}[\mathfrak{m}^{(0,2)}], v \in V_\sigma \right\}$$

because $L(\sigma)$ is a full generalized Verma module. This translates into $\xi(L(\sigma))$ consists of finite sums of functions in $\mathrm{Hol}(\widetilde{M}, \widetilde{M}^{(1,1)}; V_\sigma)$ of the form

$$g \mapsto p(\zeta(g)) \times (\xi(v)(g)) \quad (\forall g \in \widetilde{M}) \tag{17}$$

where $p \in \mathbb{C}[\mathfrak{m}^{(0,2)}]$ and $v \in V_\sigma \subset L(\sigma)$.

Let

$$\mathcal{C}(\widetilde{G}, \widetilde{K}; V_\sigma) = \left\{ f \in \mathcal{C}(\widetilde{G}, V_\sigma) \;\middle|\; f(gk) = k^{-1}f(g)\ \forall k \in \widetilde{K} \right\}$$

be the space of continuous sections. We have a map $\xi_0\colon \rho^* \to \mathcal{C}(\widetilde{G}, \widetilde{K}; V_\sigma)$ given by

$$w \mapsto \left(g \mapsto \sum_\lambda \overline{\langle \rho^*(g^{-1})w, d_\lambda^* \rangle_{\rho^*}} d_\lambda \right) \quad (\forall g \in \widetilde{G}).$$

Let $\Omega_0 \simeq \widetilde{G}/\widetilde{K}$ denote the image $\zeta(\widetilde{G})$ in Ω. We have $\mathrm{P}_0\colon \mathcal{C}(\widetilde{G}, \widetilde{K}; V_\sigma) \to \mathcal{C}(\Omega_0; V_\sigma)$ defined by the same formula as P. We denote $\overline{\xi}_0 = \mathrm{P}_0 \circ \xi_0$. There is a positive function $m(x)$ on Ω_0 such that $m(x)dx$ is a $\widetilde{G}$-invariant measure on Ω_0. Let $w \in L(\sigma)$ such that $\overline{\xi}(w) =$

$p \times \overline{\xi}(d_1)$. Let $v = d_1^*$. Then $v \otimes w \in \rho^* \otimes L(\sigma)$.

$$\begin{aligned}
\left\langle v \otimes w, \sum_\lambda d_\lambda^* \otimes d_\lambda \right\rangle_{\rho^*}^{\sigma} &= \int_G \sum_\lambda \left\langle \rho^*(g^{-1})v, d_\lambda^* \right\rangle_{\rho^*} \left\langle g^{-1}w, d_\lambda \right\rangle_{L(\sigma)} dg \\
&= \int_G \langle \xi(w)(g), \xi_0(d_1^*)(g) \rangle_{V_\sigma} dg \\
&= \int_G p(\zeta(g)) \langle (\xi(d_1)(g)), (\xi_0(d_1^*)(g)) \rangle_{V_\sigma} dg \quad \text{(By (17).)} \\
&= \int_{G/K} p(\zeta(gK)) \langle \xi(d_1)(gK), \xi_0(d_1^*)(gK) \rangle_{V_\sigma} dgK \\
&= \int_{\Omega_0} p(x) \left\langle \overline{\xi}(d_1)(x), \overline{\xi}_0(d_1^*)(x) \right\rangle_{V_\sigma} m(x)\, dx \\
&= \int_{\Omega_0} p(x) f(x) dx \qquad (18)
\end{aligned}$$

where $f(x) = \left\langle \overline{\xi}(d_1)(x), \overline{\xi}_0(d_1^*)(x) \right\rangle_{V_\sigma} m(x)$. The function $f(x)$ is a nonzero continuous function because $f(0) = \left\langle \overline{\xi}(d_1)(0), \overline{\xi}_0(d_1^*)(0) \right\rangle_{V_\sigma} m(0) = \langle d_1, d_1 \rangle m(0) = m(0) \neq 0$. We extend $f(x)$ to the boundary of Ω_0 by 0.

By Li [17], the integration (18) is absolute convergent for every $p \in \mathbb{C}[\mathfrak{m}^{(0,2)}]$. This is the place where we exclude the Case (†) in Section 1.1.

It remains to show that (18) is nonzero for some $p(x) \in \mathbb{C}[\mathfrak{m}^{(0,2)}]$. By [11], the restriction of $\mathbb{C}[\mathfrak{m}^{(0,2)}]$ to the compact subset $\overline{\Omega_0}$ forms a dense subset in $\mathcal{C}(\overline{\Omega_0})$ under sup-norm by the Stone-Weierstrass Theorem. Since $f(x)$ is nonzero, $\int_{\overline{\Omega_0}} p(x) f(x)\, dx$ is nonzero for some $p(x)$ by the dominated convergence theorem. This completes the proof of Theorem A.

3. Natural filtration and corresponding $(\mathcal{S}(\mathfrak{p}), K)$-module

3.1. Let (G, G') be an irreducible Type I dual pair as in Table 1. In this subsection, we do not assume that it is in stable range. Let ρ be an irreducible genuine $(\mathfrak{g}, \widetilde{K})$-module. Let ρ^* denote its dual (contragredient) $(\mathfrak{g}, \widetilde{K})$-module and let $\rho' = \Theta(\rho)$ denote its full theta lift. For any module ϱ, denote its underlying space by V_ϱ.

Proposition 3.2. *We have*

$$\Theta(\rho) \cong (\rho^* \otimes \mathscr{Y})_{\mathfrak{g},K} \cong ((\rho^* \otimes \mathscr{Y})_{\mathfrak{p}})^K .$$

Proof. Since K is compact, the last equality follows if we identifying K-invariant quotient as K-invariant subspace.

Now we prove the first identity. Let $\mathscr{N} = \cap_{\psi \in \mathrm{Hom}_{\mathfrak{g},K}(\mathscr{Y},\rho)} \ker \psi$ as in (1). We have

$$\begin{aligned}
&\mathrm{Hom}_{\mathbb{C}}((V_{\rho^*} \otimes \mathscr{Y})_{\mathfrak{g},K}, \mathbb{C}) = \mathrm{Hom}_{\mathfrak{g},K}((V_{\rho^*} \otimes \mathscr{Y})_{\mathfrak{g},K}, \mathbb{C}) \\
&\quad = \mathrm{Hom}_{\mathfrak{g},K}(V_{\rho^*} \otimes \mathscr{Y}, \mathbb{C}) = \mathrm{Hom}_{\mathfrak{g},\widetilde{K}}(\mathscr{Y}, \mathrm{Hom}_{\mathbb{C}}(V_{\rho^*}, \mathbb{C})) = \mathrm{Hom}_{\mathfrak{g},\widetilde{K}}(\mathscr{Y}, V_\rho) \\
&\quad = \mathrm{Hom}_{\mathfrak{g},\widetilde{K}}(\mathscr{Y}/\mathscr{N}, V_\rho). \qquad (19)
\end{aligned}$$

The second last equality follows from the fact that $\mathscr{Y}$ is $\widetilde{K}$-finite and $\mathrm{Hom}_{\mathbb{C}}(V_{\rho^*}, \mathbb{C})_{\widetilde{K}-\text{finite}} = V_\rho$. Starting from (19), we reverse the steps by replacing $\mathscr{Y}$ with $\mathscr{Y}/\mathscr{N}$ in (1) and we get

$$\begin{aligned}
&\mathrm{Hom}_{\mathbb{C}}((V_{\rho^*} \otimes \mathscr{Y})_{\mathfrak{g},K}, \mathbb{C}) \cong \mathrm{Hom}_{\mathbb{C}}((V_{\rho^*} \otimes \mathscr{Y}/\mathscr{N})_{\mathfrak{g},K}, \mathbb{C}) \\
&\cong \mathrm{Hom}_{\mathbb{C}}((V_{\rho^*} \otimes V_\rho \otimes V_{\rho'})_{\mathfrak{g},K}, \mathbb{C}) \cong \mathrm{Hom}_{\mathbb{C}}(V_{\rho'}, \mathbb{C})
\end{aligned}$$

This proves the proposition. □

3.3. The Fock model $\mathscr{Y}$ is realized as complex polynomials on W, so $\mathscr{Y} = \cup \mathscr{Y}_b$ is filtered by degrees. Let $q : \mathscr{Y} \to V_\rho \otimes V_{\rho'}$ be the natural quotient map. Let (τ, V_τ) be a lowest degree $\widetilde{K}$-type of (ρ, V_ρ) of degree j_0. Let $V_\tau \otimes V_{\tau'}$ be the joint harmonics. We define a filtration on $V_{\rho'} = \cup_j V'_j$ by $V'_j = \mathcal{U}_j(\mathfrak{g}')V_{\tau'}$.

We view $V_{\rho^*} = \mathrm{Hom}_{\mathbb{C}}(V_\rho, \mathbb{C})_{\widetilde{K}-\text{finite}}$. Let $V_{\tau^*} \subset V_{\rho^*}$ be an irreducible $\widetilde{K}$-submodule with type τ^* which pairs perfectly with V_τ. By Theorem 13 (5) in [10], the lowest degree $\widetilde{K}$-type has multiplicity one in ρ. Hence V_τ and V_{τ^*} are well defined.

Let $l \in V_{\tau^*}$ be a nonzero linear functional on V_ρ. We identify $V_{\rho'}$ with $l(q(\mathscr{Y}))$. Then we have shown in a previous paper [19] that $l(q(\mathscr{Y}_{j_0+j})) = V'_{\lfloor j/2 \rfloor}$. Likewise we define a filtration on V_ρ and V_{ρ^*} by $\{\, V_j := \mathcal{U}_j(\mathfrak{g})V_\tau \,\}$ and $\{\, V^*_j := \mathcal{U}_j(\mathfrak{g})V_{\tau^*} \,\}$ respectively. We will clarify the relationships between them in Appendix B.4.

We define a filtration on $\mathbf{W} = V_{\rho^*} \otimes \mathscr{Y}$ by

$$\mathbf{W}_j = \sum_{2a+b=j} V^*_a \otimes \mathscr{Y}_b. \tag{20}$$

By Proposition 3.2, we have a quotient map $\eta\colon \mathbf{W} \to \mathbf{W}_{\mathfrak{g},K} = \rho'$ and it induces a filtration $F'_j = \eta(\mathbf{W}_{j_0+2j})$ on ρ'.

Lemma 3.4. *The filtrations F'_j and V'_j on $(\rho', V_{\rho'})$ are the same.*

Proof. We consider the maps

$$\begin{tikzcd} \rho^* \otimes \mathscr{Y} \arrow[r, two heads, "\mathrm{pr}_{\mathfrak{p}}"] \arrow[rr, bend left, two heads, "\eta"] & (\rho^* \otimes \mathscr{Y})_{\mathfrak{p}} \arrow[r, two heads, "\mathrm{pr}_K"] & ((\rho^* \otimes \mathscr{Y})_{\mathfrak{p}})^K = \rho'. \end{tikzcd}$$

Since $l(q'(\mathscr{Y}_{j_0+j})) = V'_{\lfloor j/2 \rfloor}$, we have $\eta(V_{\tau^*} \otimes \mathscr{Y}_{j_0+2j}) = V'_j$. Hence $V'_j \subseteq F'_j$. On the other hand, for $a + \lfloor b/2 \rfloor = j$,

$$\begin{aligned} \eta(V^*_a \otimes \mathscr{Y}_{j_0+b}) &= \eta(\mathcal{U}_a(\mathfrak{g})V_{\tau^*} \otimes \mathscr{Y}_{j_0+b}) \\ &= \eta(V_{\tau^*} \otimes \mathcal{U}_a(\mathfrak{g})\mathscr{Y}_{j_0+b}) \quad \text{(Since } \mathfrak{g} \text{ acts trivially on the image)} \\ &\subseteq \eta(V_{\tau^*} \otimes \mathscr{Y}_{j_0+b+2a}) \subseteq V'_j. \end{aligned}$$

By (20), $F'_j \subseteq V'_j$ and this proves the lemma. ☐

Taking the graded module, η induces a map[1]

$$\mathrm{Gr}\, \rho^* \otimes_{\mathcal{S}(\mathfrak{p})} \mathrm{Gr}\, \mathscr{Y} \twoheadrightarrow \mathrm{Gr}\, (\mathrm{pr}_{\mathfrak{p}}(\mathbf{W})) \xrightarrow{\mathrm{Gr}\,\mathrm{pr}_K} \mathrm{Gr}\, \rho'. \tag{21}$$

We recall that U is a maximal compact subgroup of $\mathrm{Sp}(W_{\mathbb{R}})$. Let $\mathfrak{sp}(W_{\mathbb{R}}) \otimes \mathbb{C} = \mathfrak{u} \oplus \mathfrak{s}$ denote the complexified Cartan decomposition. We recall the minimal one dimensional $\widetilde{\mathrm{U}}$-type ς of the Fock model $\mathscr{Y}$. For $k \in \mathrm{U}$, $\varsigma^{-2}(k)$ is equal to the determinant of the k action on W. We extend ς to a $(\mathcal{S}(\mathfrak{s}), \widetilde{\mathrm{U}})$-module where $\mathfrak{s}$ acts trivially. We will continue to denote this one dimensional module by ς. In this way, $\mathrm{Gr}\, \mathscr{Y} = \oplus(\mathscr{Y}_{a+1}/\mathscr{Y}_a) \simeq \varsigma \cdot \mathbb{C}[W]$ where U acts geometrically on $\mathbb{C}[W]$. Since (G, G') is a reductive dual pair in $\mathrm{Sp}(W_{\mathbb{R}})$, we denote the restriction of ς as a $(\mathcal{S}(\mathfrak{p}), \widetilde{K})$-module by $\varsigma|_{\widetilde{K}}$. Similarly we get a one dimensional $(\mathcal{S}(\mathfrak{p}'), \widetilde{K}')$-module $\varsigma|_{\widetilde{K}'}$.

Let $\mathbf{A} = \varsigma|_{\widetilde{K}} \mathrm{Gr}\, V_{\rho^*}$ and $\mathbf{B} = \varsigma|^{-1}_{\widetilde{K}'} \mathrm{Gr}\, V_{\rho'}$. Since ρ is a genuine Harish-Chandra module of $\widetilde{G}$, $\mathbf{A}$ is an $(\mathcal{S}(\mathfrak{p}), K_{\mathbb{C}})$-module. Similarly $\mathbf{B}$ is an $(\mathcal{S}(\mathfrak{p}'), K'_{\mathbb{C}})$-module.

[1]Warning: By definition $X \in \mathfrak{p}$ acts on $\mathrm{Gr}\, \rho^*$ by $-X$, since $X \times 1 = -1 \times X$ for $X \in \mathfrak{p}$ in the image. Note that there is a K-equivariant algebra isomorphism $\epsilon\colon \mathcal{S}(\mathfrak{p}) \to \mathcal{S}(\mathfrak{p})$ extending $X \mapsto -X$. It's easy to see that $\epsilon^*(\mathrm{Gr}\, \rho^*) \to \mathrm{Gr}\, \rho^*$ given by $a \mapsto (-1)^j a$ for $a \in \mathrm{Gr}_j\, \rho^*$ is an isomorphism of $\mathcal{S}(\mathfrak{p})$-modules.

We take into account ς, and the fact that K acts on $\mathbf{A}\otimes\mathbb{C}[W]$ reductively and preserves the degrees. Then (21) gives the following $(\mathcal{S}(\mathfrak{p}'),K')$-module morphisms

$$\mathbf{A}\otimes_{\mathcal{S}(\mathfrak{p})}\mathbb{C}[W]\xrightarrow{\eta_1}\left(\mathbf{A}\otimes_{\mathcal{S}(\mathfrak{p})}\mathbb{C}[W]\right)^K\xrightarrow{\eta_0}\varsigma|_{\widetilde{K}'}^{-1}\left(\mathrm{Gr}\left(\mathrm{pr}_{\mathfrak{p}}(\mathbf{W})\right)\right)^K\cong\mathbf{B}. \tag{22}$$

The merit of introducing ς is that the $\widetilde{K}\cdot\widetilde{K}'$ action on $\mathrm{Gr}\,\mathscr{Y}$ descends to a geometric $K_{\mathbb{C}}\cdot K'_{\mathbb{C}}$ action on $\mathbb{C}[W]$.

Lemma 3.5. *Suppose (G,G') is in the stable range where G is the smaller member. Then η_0 in (22) is an isomorphism, i.e.*

$$\mathbf{B}=(\mathbf{A}\otimes_{\mathcal{S}(\mathfrak{p})}\mathbb{C}[W])^{K_{\mathbb{C}}}$$

as $(\mathcal{S}(\mathfrak{p}'),K'_{\mathbb{C}})$-modules where $\mathcal{S}(\mathfrak{p}')$ and $K'_{\mathbb{C}}$ act on $\mathbb{C}[W]$.

Proof. We recall the harmonics $\widetilde{\mathcal{H}}$ in (11). Let $\mathcal{H}=\varsigma^{-1}\widetilde{\mathcal{H}}$. Under the stable range assumption $\mathbb{C}[W]=\mathcal{S}(\mathfrak{p})\otimes\mathcal{H}$ as an $(\mathcal{S}(\mathfrak{p}),K)\times K'$-module. Since $\mathbf{A}=\varsigma|_{\widetilde{K}}V_{\rho^*}$, as K'-modules,

$$(\mathbf{A}\otimes_{\mathcal{S}(\mathfrak{p})}\mathbb{C}[W])^K=(\mathbf{A}\otimes_{\mathcal{S}(\mathfrak{p})}(\mathcal{S}(\mathfrak{p})\otimes\mathcal{H}))^K=(\mathbf{A}|_K\otimes\mathcal{H})^K=\mathbf{B}|_{K'}$$

by Proposition 2.2. The map η_0 is a surjection and $\mathbf{B}$ is an admissible K'-module so the lemma follows from the equality of K'-types. $\square$

We recall the moment map in (2):

$$\mathfrak{p}^*\xleftarrow{\phi}W\xrightarrow{\phi'}\mathfrak{p}'^*.$$

For the dual pairs in Table 1, we describe explicitly the moment maps in Table 2 below. Here J_{2p} is the skew symmetric $2p$ by $2p$ matrix $\begin{pmatrix}0&1\\-1&0\end{pmatrix}$.

G	G'	W $w\in W$	$\mathfrak{p}^*$ $\phi(w)$	$\mathfrak{p}'^*$ $\phi'(w)$
$\mathrm{Sp}(2n,\mathbb{R})$	$\mathrm{O}(p,q)$	$M_{p,n}\times M_{q,n}$ (A,B)	$\mathrm{Sym}_n\times\mathrm{Sym}_n$ (A^TA,B^TB)	$M_{p,q}$ AB^T
$\mathrm{U}(n_1,n_2)$	$\mathrm{U}(p,q)$	$M_{p,n_1}\times M_{p,n_2}\times M_{q,n_1}\times M_{q,n_2}$ (A,B,C,D)	$M_{n_1,n_2}\times M_{n_2,n_1}$ (A^TB,D^TC)	$M_{p,q}\times M_{q,p}$ (AC^T,DB^T)
$\mathrm{O}^*(2n)$	$\mathrm{Sp}(p,q)$	$M_{2p,n}\times M_{2q,n}$ (A,B)	$\mathrm{Alt}_n\times\mathrm{Alt}_n$ $(A^TJ_{2p}A,B^TJ_{2q}B)$	$M_{2p,2q}$ AB^T
$\mathrm{Sp}(2n,\mathbb{C})$	$\mathrm{O}(p,\mathbb{C})$	$M_{p,2n}$ A	Sym_{2n} A^TA	Alt_p $AJ_{2n}A^T$

Table 2. Moment maps for non-compact dual pairs.

Since the filtration on ρ^* (resp. ρ') is good, the graded module $\mathbf{A}$ (resp. $\mathbf{B}$) is a finitely generated $\mathcal{S}(\mathfrak{p})$-module (resp. $\mathcal{S}(\mathfrak{p}')$-module). Let $\mathscr{A}$ (resp. $\mathscr{B}$) be the associated coherent sheaf on $\mathfrak{p}^*$ (resp. $\mathfrak{p}'^*$). The associated quasi-coherent sheaf of $\mathbf{A}\otimes_{\mathcal{S}(\mathfrak{p})}\mathbb{C}[W]$ on $\mathfrak{p}'^*$ is $\phi'_*\phi^*\mathscr{A}$. Let $\mathscr{B}$ be the associated quasi-coherent sheaf of $\mathbf{B}$ on $\mathfrak{p}'^*$.

3.6. *Proof of Theorem B.* By definition, $\mathrm{AV}(\rho^*)=\mathrm{Supp}(\mathscr{A})$ and $\mathrm{AV}(\Theta(\rho))=\mathrm{Supp}(\mathscr{B})$. By (22), $\mathscr{B}$ is a subquotient of the quasi-coherent sheaf $\phi'_*\phi^*\mathscr{A}$ so

$$\mathrm{Supp}(\mathscr{B})\subseteq\mathrm{Supp}(\phi'_*\phi^*\mathscr{A})\subseteq\overline{\phi'(\mathrm{Supp}(\phi^*\mathscr{A}))}\subseteq\overline{\phi'(\phi^{-1}(\mathrm{Supp}(\mathscr{A})))}=\theta(\mathrm{Supp}(\mathscr{A})).$$

This proves the theorem. $\square$

The above proof also applies to Type II reductive dual pairs. If $\mathrm{AV}(\rho)$ is irreducible, then the above proposition is Proposition 3.12 in [25].

4. Associated cycles

Throughout this section, we suppose (G, G') is in the stable range where G is the smaller member. Let ρ be an irreducible genuine $(\mathfrak{g}, \widetilde{K})$-module. The objective of this section is to prove Theorems C and D. For a point or a subvariety Z of $\mathfrak{p}^*$, $\mathfrak{p}'^*$ or W, we let i_Z denote the natural inclusion map.

4.1. We will first state and prove Lemma 4.2. This is a key lemma which could be view as an enhance of Section 1.3 in [25]. One may skip its proof in the first reading.

Let $\mathcal{O} \in \mathfrak{N}_{K_\mathbb{C}}(\mathfrak{p}^*)$ and $\mathcal{O}' = \theta(\mathcal{O})$. Pick a $w \in W$ such that $x = \phi(w) \in \mathcal{O}$ and $x' = \phi'(w) \in \mathcal{O}'$. Let $\alpha\colon K'_{x'} \to K_x$ be the map defined by (35). Then pre-composition with α defines a map α^* from the set of K_x-modules (resp. characters of K_x) to the set of $K'_{x'}$-modules (resp. virtual characters of $K'_{x'}$).

Lemma 4.2. *Let A be a $(\mathbb{C}[\overline{\mathcal{O}}], K_\mathbb{C})$-module. Define an $(\mathcal{S}(\mathfrak{p}'), K'_\mathbb{C})$-module by*

$$B = (\mathbb{C}[W] \otimes_{\mathcal{S}(\mathfrak{p})} A)^{K_\mathbb{C}}.$$

Then B is a $(\mathbb{C}[\overline{\mathcal{O}'}], K'_\mathbb{C})$-module.

Let $\mathscr{A}$ and $\mathscr{B}$ be the quasi-coherent sheaves on $\overline{\mathcal{O}}$ and $\overline{\mathcal{O}'}$ associated to A and B respectively. Then we have the following isomorphism of $K'_{x'}$-modules:

$$i^*_{x'}\mathscr{B} \cong \alpha^*(i^*_x \mathscr{A}).$$

In particular, $\dim i^*_{x'}\mathscr{B} = \dim i^*_x \mathscr{A}$ if A is finitely generated.

Proof. See [20]. Let $Z = \phi^{-1}(\overline{\mathcal{O}})$ be the set theoretic fiber. We consider following diagram

$$\begin{tikzcd}
\{ x \} \arrow[dd, hook, "i_x"'] & \{ w \} \arrow[l, "\cong"'] \arrow[d, hook, "i_w"] & \\
 & Z_{x'} \arrow[r] \arrow[d, hook, "i_{Z_{x'}}"] & \{ x' \} \arrow[d, hook, "i_{x'}"] \\
\overline{\mathcal{O}} \arrow[d, hook, "i_{\overline{\mathcal{O}}}"] & Z \arrow[l, two heads, "\phi|_Z"'] \arrow[r, "\phi'|_Z"] \arrow[d, hook, "i_Z"] & \overline{\mathcal{O}'} \arrow[d, hook, "i_{\overline{\mathcal{O}'}}"] \\
\mathfrak{p}^* & W \arrow[l, "\phi"] \arrow[r, "\phi'"'] & \mathfrak{p}'^*
\end{tikzcd}$$

By Lemma A.8, $\mathbb{C}[Z] = \mathbb{C}[W] \otimes_{S(\mathfrak{p})} \mathbb{C}[\overline{\mathcal{O}}]$. Then

$$B = (\mathbb{C}[W] \otimes_{\mathcal{S}(\mathfrak{p})} A)^{K_\mathbb{C}} = (\mathbb{C}[W] \otimes_{S(\mathfrak{p})} \mathbb{C}[\overline{\mathcal{O}}] \otimes_{\mathbb{C}[\overline{\mathcal{O}}]} A)^{K_\mathbb{C}} = (\mathbb{C}[Z] \otimes_{\mathbb{C}[\overline{\mathcal{O}}]} A)^{K_\mathbb{C}}$$

as an $(\mathcal{S}(\mathfrak{p}'), K'_\mathbb{C})$-module. By Lemma A.9, $\mathbb{C}[\overline{\mathcal{O}'}] = \mathbb{C}[Z]^{K_\mathbb{C}}$ so B is a $\mathbb{C}[\overline{\mathcal{O}'}]$-module.

Recall $x' \in \mathcal{O}'$. Let $Z_{x'} = Z \times_{\overline{\mathcal{O}'}} \{ x' \}$ be the scheme theoretical fiber. Since we are in characteristic zero, $Z_{x'}$ is reduced. Let $m(x')$ be maximal ideal in $S(\mathfrak{p}')$ corresponding to the point x'.

Since taking $K_\mathbb{C}$-invariant is an exact functor and $\phi'^*(\mathcal{S}(\mathfrak{p}'))$ is $K_\mathbb{C}$-invariant, we have

$$\begin{aligned} i_{x'}^*\mathscr{B} &= (\mathcal{S}(\mathfrak{p}')/m(x')) \otimes_{\mathcal{S}(\mathfrak{p}')} (\mathbb{C}[Z] \otimes_{\mathbb{C}[\overline{\mathcal{O}}]} A)^{K_\mathbb{C}} \\ &= (\mathbb{C}[Z] \otimes_{\mathbb{C}[\overline{\mathcal{O}}]} A)^{K_\mathbb{C}}/(m(x')\mathbb{C}[Z] \otimes_{\mathbb{C}[\overline{\mathcal{O}}]} A)^{K_\mathbb{C}} \\ &= \Big(\mathbb{C}[Z] \otimes_{\mathbb{C}[\overline{\mathcal{O}}]} A)/(m(x')\mathbb{C}[Z] \otimes_{\mathbb{C}[\overline{\mathcal{O}}]} A)\Big)^{K_\mathbb{C}} \\ &= \Big((\mathbb{C}[Z]/m(x')\mathbb{C}[Z]) \otimes_{\mathbb{C}[\overline{\mathcal{O}}]} A\Big)^{K_\mathbb{C}} \\ &= (\mathbb{C}[Z_{x'}] \otimes_{\mathbb{C}[\overline{\mathcal{O}}]} A)^{K_\mathbb{C}}. \qquad (23) \end{aligned}$$

Let $\mathscr{Z} := i^*_{Z_{x'}}(\phi|_Z)^*\mathscr{A}$, then $\mathbb{C}[Z_{x'}] \otimes_{\mathbb{C}[\overline{\mathcal{O}}]} A = \mathscr{Z}(Z_{x'})$. By Corollary A.5, $Z_{x'}$ is a $K_\mathbb{C} \times K_{x'}$-orbit generated by w. Let $S_w = \mathrm{Stab}_{K_\mathbb{C}\times K'_{x'}}(w)$. By (36), $S_w = K_x \times_\alpha K'_{x'} = \{(\alpha(k'), k') \in K_x \times K'_{x'}\}$. Then by Theorem 2.7 in [3],

$$\mathscr{Z}(Z_{x'}) = \mathrm{Ind}_{K_x \times_\alpha K'_{x'}}^{K_\mathbb{C}\times K'_{x'}} \chi$$

where χ is the fiber of $\mathscr{Z}$ at w. By the above commutative diagram, we have S_w-module isomorphisms

$$\chi = i_w^* i^*_{Z_{x'}}(\phi|_Z)^*\mathscr{A} \cong i_x^*\mathscr{A}.$$

where $(\alpha(k'), k') \in S_w = K_x \times_\alpha K'_{x'}$ acts on $i_x^*\mathscr{A}$ via the natural action of $\alpha(k')$ on $i_x^*\mathscr{A}$.[2]

Putting the above into (23) gives

$$i_{x'}^*\mathscr{B} = (\mathscr{Z}(Z_{x'}))^{K_\mathbb{C}} = (\mathrm{Ind}_{K_x \times_\alpha K'_{x'}}^{K_\mathbb{C}\times K'_{x'}} \chi)^{K_\mathbb{C}} \cong \chi \circ \alpha$$

as K'_x-modules. We prove the right isomorphism: Let $f \in (\mathrm{Ind}_{K_x \times_\alpha K'_{x'}}^{K_\mathbb{C}\times K'_{x'}} \chi)^{K_\mathbb{C}}$. Then $f : K_\mathbb{C} \times K'_{x'} \to \chi$ satisfies $f(k, k') = \chi(\alpha(k'))f(\alpha(k')^{-1}k, 1) = \chi(\alpha(k'))f(1, 1)$. Hence f is uniquely determined by $f(1, 1)$. This proves the right isomorphism. It also completes the proof of the lemma. □

4.3. Let $\rho' = \Theta(\rho)$, $\mathbf{A} = \varsigma|_{\widetilde{K}} \mathrm{Gr}\, V_{\rho^*}$ and $\mathbf{B} = \varsigma|^{-1}_{\widetilde{K}'} \mathrm{Gr}\,\Theta(V_\rho)$ as before. For a subset Z of $\mathfrak{p}^*$, we let $I(Z)$ denote the ideal of $\mathcal{S}(\mathfrak{p})$ vanishing on Z.

Proposition 4.4. *There is a finite filtration* $0 = \mathbf{A}_0 \subset \cdots \subset \mathbf{A}_l \subset \mathbf{A}_{l+1} \subset \cdots \subset \mathbf{A}_n = \mathbf{A}$ *of* $(\mathcal{S}(\mathfrak{p}), K_\mathbb{C})$*-modules with the following property: For each* l*, there is a* $K_\mathbb{C}$*-orbit* $\mathcal{O}_l$ *such that the annihilator ideal of* $\mathbf{A}_l/\mathbf{A}_{l-1}$ *in* $\mathcal{S}(\mathfrak{p})$ *is the ideal* $I(\overline{\mathcal{O}_l})$*.*

In particular $\mathbf{A}_l/\mathbf{A}_{l-1}$ *is a* $\mathbb{C}[\overline{\mathcal{O}_l}]$*-module and* $\bigcup_{l=1}^n \overline{\mathcal{O}_l} = \mathrm{AV}(\rho^*)$*.*

Remark. We warn the orbit $\mathcal{O}_l$ may not be connected since $K_\mathbb{C}$ may not be connected. Furthermore a $\mathcal{O}_l$ may not be an open orbit in $\mathrm{AV}(\rho^*)$.

Proof. The proof essentially follows that of Lemma 2.11 in [30].

Let K_0 be the connected component of $K_\mathbb{C}$. The set of isolated primes of $\mathbf{A}$ is finite and discrete. The connected group K_0 acts trivially on it.

Let $a \in \mathbf{A}$ such that $\mathscr{D} = \mathrm{Ann}_{\mathcal{S}(\mathfrak{p})}(a)$ is an isolated prime of $\mathbf{A}$. Let $\mathbf{A}_1 = \mathcal{S}(\mathfrak{p})K_\mathbb{C} a$ be the $(\mathcal{S}(\mathfrak{p}), K_\mathbb{C})$-submodule in $\mathbf{A}$ generated by a. Let $V(\mathscr{D})$ be the subset of $\mathfrak{p}^*$ cut out

[2] Let $m(w)$ be the maximal ideal of w in $\mathbb{C}[Z]$ and $m(x)$ be the maximal ideal of x in $\mathbb{C}[\overline{\mathcal{O}}]$. Then the map $\phi : w \mapsto x$ gives a $\mathbb{C}[\overline{\mathcal{O}}]$-algebra isomorphism: $L : \mathbb{C}[\overline{\mathcal{O}}]/m(x) \xrightarrow{\cong} \mathbb{C}[Z]/m(w) \cong \mathbb{C}$. The group S_w acts on the left hand side while the group K_x acts on the right hand side. These two actions are compatible in the sense that for $(\alpha(k'), k') \in S_w \subset K_x \times K'_{x'}$, we have $L \circ \alpha(k') = (\alpha(k'), k') \circ L$.

Similarly $K'_\mathbb{C}$ acts on $\mathbb{C}[Z] \otimes_{\mathbb{C}[\overline{\mathcal{O}}]} A$ via translation on $\mathbb{C}[Z]$ while $K_\mathbb{C}$ acts via translation on Z and on A. Then $\chi = (\mathbb{C}[Z]/m(w)) \otimes_{\mathbb{C}[\overline{\mathcal{O}}]} A \cong (\mathbb{C}[\overline{\mathcal{O}}]/m(x)) \otimes_{\mathbb{C}[\overline{\mathcal{O}}]} \mathbf{A} = i_x^*\mathscr{A}$. Let $(\alpha(k'), k') \in S_w$. Then it acts on the right hand side via its natural action of $\alpha(k')$ on $i_x^*\mathscr{A}$.

by $\mathscr{D}$. Since $V(\mathscr{D})$ is irreducible and K_0-invariant, it is the closure of single K_0-orbit $\mathcal{O}_0$. Let $\mathcal{O}_1 = K_{\mathbb{C}}\mathcal{O}_0$.

We claim that $\operatorname{Ann}_{\mathcal{S}(\mathfrak{p})}(\mathbf{A}_1) = I(\overline{\mathcal{O}_1})$. Indeed,

$$\begin{aligned}\operatorname{Ann}_{\mathcal{S}(\mathfrak{p})}(\mathbf{A}_1) &= \bigcap_{k\in K_{\mathbb{C}}} \operatorname{Ann}_{\mathcal{S}(\mathfrak{p})}(k\cdot a) = \bigcap_{k\in K_{\mathbb{C}}} k\cdot\mathscr{D} = \bigcap_{[k]\in K_{\mathbb{C}}/K_0} [k]\cdot\mathscr{D} \\ &= I(\bigcup_{[k]\in K_{\mathbb{C}}/K_0} [k]\cdot V(\mathscr{D})) = I(\overline{\mathcal{O}_1}).\end{aligned}$$

The above second last equality holds because it is a finite intersection of prime ideals. The last equality holds by the definition of $\mathcal{O}_1$. This proves our claim.

Now, we could construct $\mathbf{A}_l$ and $\mathcal{O}_l$ inductively by applying the above construction to the $(\mathcal{S}(\mathfrak{p}), K_{\mathbb{C}})$-module $\mathbf{A}/\mathbf{A}_{l-1}$. This procedure will eventually stop because $\mathbf{A}$ is a finite generated module over Neotherian ring $\mathcal{S}(\mathfrak{p})$. $\square$

Let $\mathbf{A}_l$ as in Proposition 4.4 and let $\mathbf{A}^l = \mathbf{A}_l/\mathbf{A}_{l-1}$. It is a finitely generated module of $\mathbb{C}[\overline{\mathcal{O}_l}]$ and we let $\mathscr{A}^l$ be its associated coherent sheaf on $\overline{\mathcal{O}_l}$.

By Lemma 3.5, $\mathbf{B} = (\mathbb{C}[W]\otimes_{\mathcal{S}(\mathfrak{p})}\mathbf{A})^{K_{\mathbb{C}}}$. Hence $\mathbf{B}_l = (\mathbb{C}[W]\otimes_{\mathcal{S}(\mathfrak{p})}\mathbf{A}_l)^{K_{\mathbb{C}}}$ is a $(\mathcal{S}(\mathfrak{p}), K_{\mathbb{C}})$-equivariant filtration of $\mathbf{B}$. We set $\mathbf{B}^l = \mathbf{B}_l/\mathbf{B}_{l-1}$. By Theorem A.7 in Appendix A.6, ϕ is flat. Combining this with the exactness of taking $K_{\mathbb{C}}$-invariants, we get

$$\mathbf{B}^l = \mathbf{B}_l/\mathbf{B}_{l-1} = (\mathbb{C}[W]\otimes_{\mathcal{S}(\mathfrak{p})}(\mathbf{A}_l/\mathbf{A}_{l-1}))^{K_{\mathbb{C}}} = (\mathbb{C}[W]\otimes_{\mathcal{S}(\mathfrak{p})}\mathbf{A}^l)^{K_{\mathbb{C}}}. \tag{24}$$

Let $\mathscr{B}_l$ and $\mathscr{B}^l$ be the associated coherent sheaves of $\mathbf{B}_l$ and $\mathbf{B}^l$ respectively.

4.5. We define a partial ordering on the $K_{\mathbb{C}}$-orbits by containments in the Zariski closures. Let $\{\mathcal{O}_{l_1},\ldots,\mathcal{O}_{l_r}\}$ be the set of (distinct) maximal nilpotent $K_{\mathbb{C}}$-orbits appearing in Proposition 4.4. For each $\mathcal{O}_{l_j}$ in this set, we fix a representative $x_j\in\mathcal{O}_{l_j}$ and define a K_{x_j}-module

$$\chi(x_j,\operatorname{Gr}\mathscr{A}) = \bigoplus_{\mathcal{O}_{l_j}=\mathcal{O}_l} i_{x_j}^*\mathscr{A}^l. \tag{25}$$

Let $m_j = \dim_{\mathbb{C}}\chi(x_j,\operatorname{Gr}\mathscr{A})$. The integer m_j is independent of the choice of $x_j\in\mathcal{O}_{l_j}$. Moreover $m_j\neq 0$. Indeed all K_{x_j}-modules in the right hand side of (25) is non-zero because $\operatorname{supp}(\mathscr{A}^l) = V(\operatorname{Ann}_{\mathcal{S}(\mathfrak{p})}\mathbf{A}^l) = \overline{\mathcal{O}_l} = \overline{\mathcal{O}_{l_j}}$.

Recall that $\mathbf{A} = \varsigma|_{\widetilde{K}}\operatorname{Gr}V_{\rho^*}$. Let

$$\chi_{x_j} = \varsigma|_{\widetilde{K}}^{-1}\otimes\chi(x_j,\operatorname{Gr}\mathscr{A}). \tag{26}$$

Then $\{\,(\mathcal{O}_{l_j},x_j,\chi_{x_j})\,\}$ is the set of orbit data attached to the filtrations given by Proposition 4.4.

Now the associated cycle of ρ^* is

$$\operatorname{AC}(\rho^*) = \operatorname{AC}(\mathbf{A}) = \sum_{j=1}^r m_j[\overline{\mathcal{O}_{l_j}}]. \tag{27}$$

and the associated variety $\operatorname{AV}(\rho^*) = \cup_{l=1}^r\overline{\mathcal{O}_{l_j}}$.

4.6. *Proof of Theorems C and D.* First we observe following lemma.

Lemma 4.7. *Let $\{\,\mathcal{O}_{l_j} : j=1,\ldots,r\,\}$ be the set of all distinct (open) maximal $K_{\mathbb{C}}$-orbits in* $\operatorname{AV}(\rho^*)$*. Then $\{\,\theta(\mathcal{O}_{l_j}) : j=1,\ldots,r\,\}$ forms the set of all distinct (open) maximal $K'_{\mathbb{C}}$-orbits in* $\theta(\operatorname{AV}(\rho^*))$*.*

Proof. By Theorem A.2(i) the map $\theta\colon \mathfrak{N}_{K_{\mathbb{C}}}(\mathfrak{p}^*) \to \mathfrak{N}_{K'_{\mathbb{C}}}(\mathfrak{p}'^*)$ is injective so all the $\theta(\mathcal{O}_{l_j})$'s are distinct. We also have $\theta(\mathrm{AV}(\rho^*)) = \phi'(\phi^{-1}(\bigcup_{j=1}^r \overline{\mathcal{O}_{l_j}})) = \bigcup_{l=1}^r \theta(\overline{\mathcal{O}_{l_k}}) = \bigcup_{l=1}^r \overline{\theta(\mathcal{O}_{l_k})}$. It suffices to show that $\dim \theta(\mathcal{O}_{l_k}) = \dim \theta(\mathrm{AV}(\rho^*))$.

By Theorem 8.4 in [30], every $K_{\mathbb{C}}$-orbit $\mathcal{O}_{l_j}$ generates the same $G_{\mathbb{C}}$-orbit $\mathcal{O}_{\mathbb{C}}$ in $\mathfrak{g}^*$. Indeed $\overline{\mathcal{O}_{\mathbb{C}}}$ is the variety cut out by $\mathrm{Gr}\,(\mathrm{Ann}_{\mathcal{U}(\mathfrak{g})}(\rho^*))$.

Nilpotent $K_{\mathbb{C}}$-orbits for classical groups are parametrized by signed Young diagrams. In particular the underlying Young diagrams of different $\mathcal{O}_{l_j}$'s are the same and they have the same dimension equals to $\frac{1}{2}\dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}}$. By [26], the signed Young diagram of the orbit $\theta(\mathcal{O}_l)$ is obtained by adding a column to the signed Young diagram of $\mathcal{O}_{l_j}$. Hence every $K'_{\mathbb{C}}$-orbit $\theta(\mathcal{O}_{l_j})$ generates the same $G'_{\mathbb{C}}$-orbit $\mathcal{O}'_{\mathbb{C}}$ in $\mathfrak{g}'^*$ and has dimension equals to $\frac{1}{2}\dim_{\mathbb{C}} \mathcal{O}'_{\mathbb{C}}$. Moreover it is known that (for example by [KP] or [4]), $\overline{\mathcal{O}'_{\mathbb{C}}} = \theta_{\mathbb{C}}(\overline{\mathcal{O}_{\mathbb{C}}})$ where $\theta_{\mathbb{C}}$ was defined after (4). We have shown that $\dim \theta(\mathcal{O}_{l_k}) = \dim \theta(\mathrm{AV}(\rho^*))$ and this completes the proof of the lemma. $\square$

Let $\mathcal{O}'_l = \theta(\mathcal{O}_l)$. By Theorem B, $\mathrm{AV}(\rho') \subseteq \bigcup_{j=1}^r \theta(\overline{\mathcal{O}_{l_j}}) = \bigcup_{j=1}^r \mathcal{O}'_{l_j}$.

Now we apply Lemma 4.2 to $\mathbf{B}^l$ and we have

$$\chi(x'_j, \mathrm{Gr}\,\mathscr{B}) := \bigoplus_{\mathcal{O}_l = \mathcal{O}_{l_j}} i^*_{x'_j}\mathscr{B}^l \cong \bigoplus_{\mathcal{O}_l = \mathcal{O}_{l_j}} \alpha_j^*(i^*_{x_j}\mathscr{A}^l)$$

where x'_j and α_j are x' and α in the statement of the lemma respectively.

Since $\mathbf{A} = \varsigma|_{\widetilde{K}}\mathrm{Gr}\,V_{\rho^*}$ and $\mathbf{B} = \varsigma|_{\widetilde{K}'}^{-1}\mathrm{Gr}\,\Theta(V_\rho)$, the isotropic representation of $\Theta(\rho)$ at x'_j with respect to the filtration $\mathscr{B}_l$ is

$$\chi_{x'_j} = \varsigma|_{\widetilde{K}'} \otimes \chi(x'_j, \mathrm{Gr}\,\mathscr{B}) = \varsigma|_{\widetilde{K}'} \otimes \chi(x_j, \mathrm{Gr}\,, \mathscr{A}) \circ \alpha_j = \varsigma|_{\widetilde{K}'} \otimes (\varsigma|_{\widetilde{K}} \otimes \chi_{x_j}) \circ \alpha_j.$$

In particular, $\chi_{x'_j} \neq 0$ since $\chi_{x_j} \neq 0$. Therefore $\{(\mathcal{O}'_{l_j}, x'_j, \chi_{x'_j}) : j = 1, \ldots, r\}$ forms the set of orbit data attached to the filtration $\mathscr{B}_l$. This proves Theorem C.

Now

$$\mathrm{AC}(\Theta(\rho)) = \sum_{j=1}^r \dim \chi_{x'_j}[\overline{\mathcal{O}'_{l_j}}] = \sum_{j=1}^r m_j[\overline{\theta(\mathcal{O}_{l_j})}] = \theta(\mathrm{AC}(\rho^*)).$$

This proves Theorem D. $\square$

4.8. *Proof of Corollary E.* Let $\rho' = \Theta(\rho)$. From the proof of Lemma 4.7, $\overline{\mathcal{O}'_{\mathbb{C}}} = \theta_{\mathbb{C}}(\overline{\mathcal{O}_{\mathbb{C}}})$. This gives

$$\overline{G'_{\mathbb{C}}\mathrm{AV}(\rho')} = \overline{\mathcal{O}'_{\mathbb{C}}} = \theta_{\mathbb{C}}(\overline{\mathcal{O}_{\mathbb{C}}}) = \theta_{\mathbb{C}}(\overline{G_{\mathbb{C}}\mathrm{AV}(\rho)}) = \theta_{\mathbb{C}}(\mathrm{V}_{\mathbb{C}}(\rho)).$$

Although ρ' may not be irreducible, we claim that $\mathrm{V}_{\mathbb{C}}(\rho') = \overline{G'_{\mathbb{C}}\mathrm{AV}(\rho')}$ and this would prove the corollary. First $\mathrm{V}_{\mathbb{C}}$ is an additive map, i.e. $\mathrm{V}_{\mathbb{C}}(B) = \mathrm{V}_{\mathbb{C}}(A) \cup \mathrm{V}_{\mathbb{C}}(C)$ for every exact sequence $0 \to A \to B \to C \to 0$. This is well known to the experts, which follows by taking the graded version of

$$\mathrm{Ann}_{\mathcal{U}(\mathfrak{g}')}(A)\mathrm{Ann}_{\mathcal{U}(\mathfrak{g}')}(C) \subset \mathrm{Ann}_{\mathcal{U}(\mathfrak{g}')}(B) \subset \mathrm{Ann}_{\mathcal{U}(\mathfrak{g}')}(A) \cap \mathrm{Ann}_{\mathcal{U}(\mathfrak{g}')}(C).$$

Next let $\rho'_1, \ldots, \rho'_s$ be all the irreducible subquotients of the $(\mathfrak{g}', \widetilde{K}')$-model ρ' of finite length. Using Theorem 8.4 in [30] again,

$$\mathrm{V}_{\mathbb{C}}(\mathrm{Ann}\,\rho') = \bigcup_{k=1}^s \mathrm{V}_{\mathbb{C}}(\mathrm{Ann}\,\rho'_k) = \bigcup_{k=1}^s \overline{G'_{\mathbb{C}}\mathrm{AV}(\rho'_k)} = \overline{G'_{\mathbb{C}} \bigcup_{k=1}^s \mathrm{AV}(\rho'_k)} = \overline{G'_{\mathbb{C}}\mathrm{AV}(\rho')}.$$

This proves our claim and Corollary E. $\square$

5. The K-spectrum equation

We retain the notation in the previous sections where we assume that (G, G') is in stable range where G is the smaller member. The objective of this section is to prove Proposition 5.1 which implies Theorem F.

Let $x \in \mathcal{O}$ and let (χ_x, V_x) be a finite dimensional rational representation of K_x as in Section 1.6. By Theorem 2.7 in [3], there is an equivalent of categories between the category of rational representations of K_x and the category of $K_\mathbb{C}$-equivariant sheaf on $\mathcal{O} \simeq K/K_x$. Let $\mathscr{L}$ be the $K_\mathbb{C}$-equivariant sheaf on $\mathcal{O}$ corresponding to χ_x. We assume that $\mathscr{L}$ is generated by its global sections (c.f. (5)). Let $i_\mathcal{O} : \mathcal{O} \to \mathfrak{p}^*$ denote the inclusion map and let $\mathscr{A} = (i_\mathcal{O})_* \mathscr{L}$. We also set

$$A := \mathscr{A}(\mathfrak{p}^*) = \mathscr{L}(\mathcal{O}) = \mathrm{Ind}_{K_x}^{K_\mathbb{C}} V_x \tag{28}$$

as an $(\mathcal{S}(\mathfrak{p}), K_\mathbb{C})$-module. Then $V_x = i_x^* \mathscr{A} = A/m(x)A$ and $\mathrm{AC}(A) = (\dim V_x)[\overline{\mathcal{O}}]$.

Let $\mathcal{O}' = \theta(\mathcal{O})$. Let $w \in W$ such that $x = \phi(w) \in \mathcal{O}$, $x' = \phi'(w) \in \mathcal{O}'$ in (2). Let $\alpha \colon K'_{x'} \to K_x$ be the map defined in (35) in Appendix A. Let $(\chi'_{x'}, V_{x'}) = (\chi_x \circ \alpha, V_x)$ be the representation of $K'_{x'}$. Let $\mathscr{L}'$ be the K'-equivariant sheaf on $\mathcal{O}'$ corresponding to the representation $\chi'_{x'}$ of $K'_{x'}$. Let $i_{\mathcal{O}'} : \mathcal{O}' \to \mathfrak{p}'^*$ denote the inclusion map. We define the $(\mathcal{S}(\mathfrak{p}'), K')$-module

$$B = (A \otimes_{\mathcal{S}(\mathfrak{p})} \mathbb{C}[W])^{K_\mathbb{C}} = (A \otimes_{\mathbb{C}[\overline{\mathcal{O}}]} \mathbb{C}[\overline{\mathcal{O}}] \otimes_{\mathcal{S}(\mathfrak{p})} \mathbb{C}[W])^{K_\mathbb{C}} = (A \otimes_{\mathbb{C}[\overline{\mathcal{O}}]} \mathbb{C}[Z])^{K_\mathbb{C}}$$

as before. By the proof of Proposition 2.2 we have $(A|_K \otimes \mathcal{H})^K = B|_{K'}$ as K'-modules.

Proposition 5.1. *The sheaf $(i_{\mathcal{O}'})_* \mathscr{L}'$ is the coherent sheaf corresponding to the $(\mathcal{S}(\mathfrak{p}'), K')$-module B.*

Before we begin the proof of the above proposition, we show that the proposition implies Theorem F. Indeed by (28), $A|_K = \varsigma|_{\widetilde{K}} V_{\rho^*}|_{\widetilde{K}}$ as K-modules. Therefore

$$\begin{aligned}
\varsigma|_{\widetilde{K}'}^{-1} \otimes \rho'|_{\widetilde{K}'} &= \varsigma|_{\widetilde{K}'}^{-1} \mathrm{Gr}\,(\rho')|_{\widetilde{K}'} = (\varsigma|_{\widetilde{K}} V_{\rho^*}|_{\widetilde{K}} \otimes \mathcal{H})^K && \text{(by Proposition 2.2)} \\
&= (A|_K \otimes \mathcal{H})^K = B|_{K'} && \text{(by Proposition 2.2 again)} \\
&= (i_{\mathcal{O}'})_* \mathscr{L}'(\mathfrak{p}'^*) && \text{(by Proposition 5.1)} \\
&= \mathscr{L}'(\mathcal{O}') = \mathrm{Ind}_{K'_{x'}}^{K'_\mathbb{C}} V_{x'}.
\end{aligned}$$

This proves Theorem F.

Proof of Proposition 5.1. Let $Y := W \times_\mathfrak{p} \mathcal{O}$. By Lemma A.8, Y is a reduced scheme. We consider following diagram where $Z^\circ = (\phi')^{-1}(\mathcal{O}') \cap Y$.

$$\begin{tikzcd}
 & & Z^\circ \arrow[dl] \arrow[dr, "\phi'"] & \\
\mathcal{O} \arrow[d, "i_\mathcal{O}"] & Y \arrow[l, "\phi|_Y"'] \arrow[d, "i_Y"] \arrow[dr] & & \mathcal{O}' \arrow[dl] \\
\mathfrak{p}^* & W \arrow[l, "\phi"'] \arrow[r, "\phi'"] & \mathfrak{p}'^* &
\end{tikzcd}$$

Since ϕ is flat, Proposition 9.3 in Chapter 3 in [9] gives

$$\phi^* (i_\mathcal{O})_* \mathscr{L} = (i_Y)_* (\phi|_Y)^* \mathscr{L}$$

as sheaves on W. Let $\mathscr{B}$ be the qusi-coherent sheaf on $\mathfrak{p}'^*$ associated to B. Let $\mathscr{Q} = (\phi|_Y)^* \mathscr{L}$. By the exactness of taking $K_\mathbb{C}$-invariant, $\mathscr{B}(U) = (\mathscr{Q}(\phi'^{-1}(U) \cap Y))^{K_\mathbb{C}}$ for every open set $U \subset \mathfrak{p}'^*$.

i) $B = \mathscr{B}(\mathfrak{p}'^*) = (\mathscr{Q}(Y))^{K_\mathbb{C}} = (H^0(Y, \mathscr{Q}))^{K_\mathbb{C}}$.

ii) We recall $Z^\circ = (\phi')^{-1}(\mathcal{O}') \cap Y$. It gives

$$\mathscr{B}(\mathcal{O}') = \left(H^0(Z^\circ, \mathscr{Q})\right)^{K_\mathbb{C}}.$$

iii) From the proof of Lemma 4.2, we have

$$\mathscr{B}(\mathcal{O}') = \operatorname{Ind}_{K'_{x'}}^{K'_\mathbb{C}}(\chi_x \circ \alpha).$$

Thus the proposition is equivalent to $\mathscr{B}(\mathfrak{p}'^*) = \mathscr{B}(\mathcal{O}')$. It suffices to prove that

$$H^0(Y, \mathscr{Q}) = H^0(Z^\circ, \mathscr{Q}). \tag{29}$$

Indeed $\mathscr{L}$ is locally free on $\mathcal{O}$ so $\mathscr{Q}$ is locally free on Y. Hence

$$\operatorname{depth} \mathscr{Q}_y = \operatorname{depth} \mathscr{O}_{Y,y}$$

for any $y \in Y$. Let $\partial Z^\circ = Y - Z^\circ$. Let $H^i_{\partial Z^\circ}(\mathscr{Q})$ (resp. $\mathscr{H}^i_{\partial Z^\circ}(\mathscr{Q})$) be the cohomology group (resp., cohomology sheaf) of Y with coefficient in $\mathscr{Q}$ and support in ∂Z°. By Lemma A.10 in Appendix A, $\operatorname{codim}(Y, \partial Z^\circ) \geq 2$. By Lemma A.8(ii), Y is a normal scheme so it satisfies Serre's (S2) condition. Therefore,

$$\operatorname{depth}_y \mathscr{Q}_y = \operatorname{depth} \mathscr{O}_{Y,y} \geq \min \{ \dim \mathscr{O}_{Y,y}, 2 \} = 2$$

for all $y \in \partial Z^\circ$. By a vanishing theorem of Grothendieck (see [7, Theorem 3.8]),

$$\mathscr{H}^i_{\partial Z^\circ}(\mathscr{Q}) = 0 \quad \text{for} \quad i = 0, 1.$$

By Proposition 1.11 in [7], a spectral sequence argument implies

$$H^0(Y, \mathscr{Q}) \cong H^0(Z^\circ, \mathscr{Q}).$$

This completes the proof of Proposition 5.1. □

6. Admissible data

In this section we will show that the theta lift of an admissible data is still an admissible data. We continue to assume that (G, G') is a dual pair in stable range where G is the smaller member.

Let $\mathcal{O}$ be a nilpotent $K_\mathbb{C}$-orbit in $\mathfrak{p}^*$ as in (2). Let $\mathcal{O}' = \theta(\mathcal{O})$. Let $w \in W$ such that $x = \phi(w) \in \mathcal{O}$ and $x' = \phi'(w) \in \mathcal{O}'$. Let $\alpha \colon K'_{x'} \to K_x$ be the map defined by (35). Proposition G follows from the next proposition.

Proposition 6.1. *Suppose (G, G') is in stable range where G is the smaller member. Let χ_x be an admissible representation of $\widetilde{K}_x$ as defined in Section 1.7. We set*

$$\chi_{x'} := \varsigma|_{\widetilde{K}'_{x'}} \otimes (\varsigma|_{\widetilde{K}_x} \otimes \chi_x) \circ \alpha.$$

Then $\chi_{x'}$ is an admissible representation of $\widetilde{K}'_{x'}$.

Proof. We have to verify that

$$\chi_{x'}(\exp(X')) = \det(\operatorname{Ad}(\exp(X'/2))|_{(\mathfrak{k}'/\mathfrak{k}'_{x'})^*}) \qquad \forall X' \in \mathfrak{k}'_{x'}.$$

Since χ_x is admissible, by taking square of above equation, it is easy to see that it reduces to the following lemma.

Lemma 6.2. *As $\mathfrak{k}'_{x'}$-modules,*

$$\textstyle\bigwedge^{\text{top}} (\mathfrak{k}'/\mathfrak{k}'_{x'}) = (\bigwedge^{\text{top}} (\mathfrak{k}/\mathfrak{k}_x) \circ \alpha) \otimes \varsigma|_{K'}^{-2} \otimes (\varsigma|_K^{-2} \circ \alpha).$$

Proof. Let $E = K'_{\mathbb{C}} K_{\mathbb{C}} w$, $F = \phi^{-1}(x)$, and $F' = \phi^{-1}(x')$. Let $S_w := \mathrm{Stab}_{K'_{\mathbb{C}} \times K_{\mathbb{C}}}(w) = \{ (k', \alpha(k') \mid k' \in K'_{x'} \} \cong K'_{x'}$. We have following two exact sequences of S_w-modules:

$$0 \longrightarrow \mathrm{T}_w F' \longrightarrow \mathrm{T}_w E \longrightarrow \mathrm{T}_{x'} \mathcal{O}' \longrightarrow 0$$

and

$$0 \longrightarrow \mathrm{T}_w F \longrightarrow \mathrm{T}_w E \longrightarrow \mathrm{T}_x \mathcal{O} \longrightarrow 0 \; .$$

Here S_w acts on $\mathrm{T}_x \mathcal{O}$ (resp. $\mathrm{T}_{x'} \mathcal{O}'$) via the projection $S_w \to K_x$ (resp. $S_w \xrightarrow{\sim} K'_{x'}$). Since $S_w \cong K'_{x'}$, the above are also exact sequences of $K'_{x'}$-modules.

By Corollary A.5(i) $\mathrm{T}_w F' \cong \mathfrak{k}$. The $\mathfrak{k}'_{x'}$-action on $\bigwedge^{\mathrm{top}} \mathfrak{k}$ is trivial since $\mathfrak{k}$ is reductive. Therefore

$$\bigwedge\nolimits^{\mathrm{top}} \mathrm{T}_{x'} \mathcal{O}' \cong \bigwedge\nolimits^{\mathrm{top}} \mathrm{T}_x \mathcal{O} \otimes \bigwedge\nolimits^{\mathrm{top}} \mathrm{T}_w F \tag{30}$$

as $\mathfrak{k}'_{x'}$-modules. Since we are in the stable range, $\phi : W \to \mathfrak{p}^*$ is a submersion at every point $w \in W$. We have following exact sequence of $K'_{x'}$-modules:

$$0 \longrightarrow \mathrm{T}_w F \longrightarrow \mathrm{T}_w W \longrightarrow \mathrm{T}_x \mathfrak{p}^* \longrightarrow 0.$$

Since $\mathrm{T}_x \mathfrak{p}^* \cong \mathfrak{p}^*$ and $\mathfrak{k}'_{x'}$ acts trivially on $\bigwedge^{\mathrm{top}} \mathfrak{p}^*$, we have

$$\bigwedge\nolimits^{\mathrm{top}} \mathrm{T}_w W \cong \bigwedge\nolimits^{\mathrm{top}} \mathrm{T}_w F. \tag{31}$$

Combining (30), (31), $\mathrm{T}_w W \cong W$, $\mathrm{T}_x \mathcal{O} \cong \mathfrak{k}/\mathfrak{k}_x$ and $\mathrm{T}_{x'} \mathcal{O}' \cong \mathfrak{k}'/\mathfrak{k}'_{x'}$, we have

$$\bigwedge\nolimits^{\mathrm{top}} (\mathfrak{k}'/\mathfrak{k}'_{x'}) = (\bigwedge\nolimits^{\mathrm{top}} \mathfrak{k}/\mathfrak{k}_x \circ \alpha) \otimes \bigwedge\nolimits^{\mathrm{top}} W. \tag{32}$$

If we view $u \in \mathrm{U}_{\mathbb{C}}$ as a linear transformation on W, then

$$\varsigma^{-2}(u) = \det(u|_W).$$

Hence the action of $k' \in K'_{x'}$ on $\bigwedge^{\mathrm{top}} W$ is

$$\det((k', \alpha(k'))|_W) = \varsigma|_{K'}^{-2}(k') \otimes (\varsigma|_K^{-2} \circ \alpha(k')).$$

Putting this into (32) proves the lemma and Proposition 6.1. $\square$

Appendix A. Geometry of theta lifts of nilpotent orbits

Throughout this appendix, we let (G, G') denote a Type I irreducible reductive dual pair in $\mathrm{Sp}(W_{\mathbb{R}})$ in stable range where G is the smaller member as in Table 1.

A.1. We recall the moment maps defined in (2). The following fact is true for every reductive dual pairs, not necessarily in stable range. The moment map factors through the affine quotient:

$$\begin{tikzcd} W \arrow[r, two heads] \arrow[rr, bend left, "\phi'"] & W/K_{\mathbb{C}} \arrow[r, hook, "i_{W/K_{\mathbb{C}}}"] & \mathfrak{p}'^*. \end{tikzcd}$$

By the First Fundamental Theorem of classical invariant theory, $\mathbb{C}[W]^{K_{\mathbb{C}}}$ is a quotient of $\mathcal{S}(\mathfrak{p}')$, i.e. $i_{W/K_{\mathbb{C}}}$ is a closed embedding. For every $K_{\mathbb{C}}$-invariant closed subset $E \subset W$, its image in $W/K_{\mathbb{C}}$ is closed by Corollary 4.6 in [27]. This implies that $\phi'(E)$ is closed in $\mathfrak{p}'^*$. Hence for every $K_{\mathbb{C}}$-invariant subset $S \in \mathfrak{p}^*$, $\theta(S) := \phi'(\phi^{-1} S)$ is a $K'_{\mathbb{C}}$-invariant closed subset of $\mathfrak{p}'^*$.

We define the null cone of $\mathfrak{p}^*$ under $K_{\mathbb{C}}$-action to be

$$N(\mathfrak{p}^*) = \left\{ \, x \in \mathfrak{p}^* \;\middle|\; 0 \in \overline{K_{\mathbb{C}} \cdot x} \, \right\}.$$

Let $\mathfrak{N}_{K_{\mathbb{C}}}(\mathfrak{p}^*)$ be the set of nilpotent $K_{\mathbb{C}}$-orbits in $\mathfrak{p}^*$. Define $N(\mathfrak{p}'^*)$ and $\mathfrak{N}_{K'_{\mathbb{C}}}(\mathfrak{p}'^*)$ in the same way. It is well known that $\theta(S) \subset N(\mathfrak{p}'^*)$ if $S \subset N(\mathfrak{p}^*)$.

We summarize some results in [26], [4] and [24].

Theorem A.2. *Let (G, G') be a reductive dual pair in stable range as in Table 1.*

(i) For any nilpotent $K_{\mathbb{C}}$-orbit $\mathcal{O}$ in $\mathfrak{p}^*$, there is a nilpotent $K'_{\mathbb{C}}$-orbit $\mathcal{O}'$ in $\mathfrak{p}'^*$ such that

$$\phi'(\phi^{-1}(\overline{\mathcal{O}})) = \overline{\mathcal{O}'}.$$

This defines an injective map $\theta\colon \mathfrak{N}_{K_{\mathbb{C}}}(\mathfrak{p}^*) \to \mathfrak{N}_{K'_{\mathbb{C}}}(\mathfrak{p}'^*)$ given by $\mathcal{O} \mapsto \mathcal{O}'$. This map is called the theta lifts of nilpotent orbits.

(ii) Theta lifting of nilpotent orbits preserves closure relation, i.e. if $\mathcal{O}_0 \subset \overline{\mathcal{O}}$ then $\theta(\mathcal{O}_0) \subset \overline{\theta(\mathcal{O})}$. □

We refer to Table 2 where W is written as a product of matrices. Let W° be the open dense subset of elements in W whose every component has full rank. Before we discuss the finer structures of orbits, we state following lemma.

Lemma A.3. *Let (G, G') be a reductive dual pair in stable range as in Table 1.*

(i) We have $\phi'^{-1}(\phi'(W^\circ)) = W^\circ$.

(ii) For any $x' \in \phi'(W^\circ)$, $\phi'^{-1}(x') \cap W^\circ$ is a single $K_{\mathbb{C}}$-orbit where $K_{\mathbb{C}}$ acts freely.

(iii) For any $x \in \phi(W^\circ)$, $\phi^{-1}(x) \cap W^\circ$ is a single $K'_{\mathbb{C}}$-orbit.

(iv) We have one-to-one correspondences of the following sets of orbits

$$\begin{array}{ccccc}
\{\, K_{\mathbb{C}}\text{-orbits in } \phi(W^\circ) \,\} & \leftrightarrow & \{\, K_{\mathbb{C}} \times K'_{\mathbb{C}}\text{-orbits in } W^\circ \,\} & \leftrightarrow & \{\, K'_{\mathbb{C}}\text{-orbits in } \phi'(W^\circ) \,\} \\
\phi(C) & \leftarrow\!\shortmid & C & \mapsto & \phi'(C) \\
\mathcal{O} & \mapsto & \phi^{-1}(\mathcal{O}) \cap W^\circ & & \\
 & & \phi^{-1}(\mathcal{O}') = \phi^{-1}(\mathcal{O}') \cap W^\circ & \leftarrow\!\shortmid & \mathcal{O}'.
\end{array}$$

Proof. The proof for each dual pair is similar so we will give the proof for the first pair in Table 1 and leave the other cases to the reader.

Consider $(G, G') = (\mathrm{Sp}(2n, \mathbb{R}), \mathrm{O}(p, q))$, $W = M_{p,n} \times M_{q,n}$, $\mathfrak{p}'^* = M_{p,q}$ and $p, q \geq 2n$. For $(A, B) \in M_{p,n} \times M_{q,n} = W$, $\phi'(A, B) = AB^T$ has rank n if and only if A and B have rank n. This proves (i).

Let $x' \in \phi'(W^\circ)$. Let $(A, B), (A', B') \in \phi'^{-1}(x') \cap W^\circ$. We have

$$AB^T = \phi'(A, B) = x' = \phi'(A', B') = A'(B')^T. \tag{33}$$

Here x', A, B, A' and B' are all rank n matrices. Since the column space of A (resp. A') is same as the column space of x', we may assume that $A = A'$ by the action of $K_{\mathbb{C}} = \mathrm{GL}(n, \mathbb{C})$. If we interpret $A : \mathbb{C}^n \to \mathbb{C}^p$ as an injective linear map, it is clear that (33) implies $B^T = B'^T$. This proves that (A, B) and (A', B') are in the same $K_{\mathbb{C}}$-orbit. This proves $\phi'^{-1}(x') \cap W^\circ$ is a single $K_{\mathbb{C}}$-orbit.

Next suppose $k \in K_{\mathbb{C}}$ stabilizes (A, B). Hence $Ak^T = A$. Since A is an injective map, $k = \mathrm{id}$. This shows that the $K_{\mathbb{C}}$-action is faithful. This proves (ii).

Let $x \in \phi(W^\circ)$. Let $(A, B), (A', B') \in \phi^{-1}(x) \cap W^\circ$. We have

$$(A^T A, B^T B) = \phi(A, B) = x = \phi(A', B') = (A'^T A', B'^T B'). \tag{34}$$

Since $\mathrm{Ker}\, A = \mathrm{Ker}\, A' = 0$, there is an $o \in \mathrm{O}(p, \mathbb{C})$ such that $A = oA'$ by Witt's theorem (for example, see [13, Theorem 3.7.1]). The same argument applies to B and B'. Hence $\phi^{-1}(x) \cap W^\circ$ is a single orbit of $K'_{\mathbb{C}} = \mathrm{O}(p, \mathbb{C}) \times \mathrm{O}(q, \mathbb{C})$. This proves (iii).

Part (iv) follows from (i), (ii) and (iii). □

Theorem A.4. *Let (G, G') be a reductive dual pair in stable range where G is the smaller member as in Table 1. Let $\mathcal{O}' = \theta(\mathcal{O})$. Then*

(i) $\phi'^{-1}(\mathcal{O}') = \phi^{-1}(\mathcal{O}) \cap W^\circ = \phi'^{-1}(\mathcal{O}') \cap \phi^{-1}(\overline{\mathcal{O}})$ is a single $K_{\mathbb{C}} \times K'_{\mathbb{C}}$-orbit.

(ii) $\phi(\phi'^{-1}(\mathcal{O}')) = \mathcal{O}$.

The orbits in $\phi^{-1}(\mathcal{O})$ and $\phi'(\phi^{-1}(\mathcal{O}))$ could been calculated explicitly in all the cases (for example from Table 4 in [4]). Then (i) and (ii) could be checked directly from these data.

We will sketch a simpler proof below. However, we could not avoid to use following fact: $\phi^{-1}(\overline{\mathcal{O}})$ has a unique open dense $K_{\mathbb{C}} \times K'_{\mathbb{C}}$-orbit (see Theorem 2.5 in [24] for a proof via deformation argument).

Sketch of proof of Theorem A.4. For every $\mathcal{O} \in \mathfrak{N}_{K_{\mathbb{C}}}(\mathfrak{p}^*)$, Lemma A.3(iv) shows that $\phi^{-1}(\mathcal{O}) \cap W^\circ$ is a $K_{\mathbb{C}} \times K'_{\mathbb{C}}$-orbit and $\mathcal{O}' = \phi'(\phi^{-1}(\mathcal{O}) \cap W^\circ)$ is a $K'_{\mathbb{C}}$-orbit. By Lemma A.3(iv) again, $\phi^{-1}(\mathcal{O}) \cap W^\circ = \phi'^{-1}(\mathcal{O}') \cap W^\circ = \phi'^{-1}(\mathcal{O}')$. Since $\phi^{-1}(\mathcal{O}) \cap W^\circ$ is open and nonempty in $\phi^{-1}(\overline{\mathcal{O}})$, it is the unique open dense $K_{\mathbb{C}} \times K'_{\mathbb{C}}$-orbit. Hence $\overline{\mathcal{O}'} \supseteq \phi'(\overline{\phi^{-1}(\mathcal{O}) \cap W^\circ}) = \phi'(\phi^{-1}(\overline{\mathcal{O}})) \supseteq \mathcal{O}'$. This shows that $\mathcal{O}' = \theta(\mathcal{O})$. Clearly $\phi'^{-1}(\mathcal{O}') \cap \phi^{-1}(\overline{\mathcal{O}})$ is a non-empty $K_{\mathbb{C}} \times K'_{\mathbb{C}}$-invariant subset of the $K_{\mathbb{C}} \times K'_{\mathbb{C}}$-orbit $\phi'^{-1}(\mathcal{O}')$. This proves $\phi'^{-1}(\mathcal{O}') = \phi'^{-1}(\mathcal{O}') \cap \phi^{-1}(\overline{\mathcal{O}})$. This gives (i).

By Lemma A.3(iv), $\phi'^{-1}(\mathcal{O}') = \phi'^{-1}(\mathcal{O}') \cap W^\circ$ is a single $K_{\mathbb{C}} \times K'_{\mathbb{C}}$-orbit. Hence $\mathcal{O} = \phi(\phi'^{-1}(\mathcal{O}') \cap W^\circ) = \phi(\phi'^{-1}(\mathcal{O}'))$. This proves (ii). □

Corollary A.5. *Let (G, G') and $\mathcal{O}' = \theta(\mathcal{O})$ as in Theorem A.4. We fix a $w \in W$ such that $x = \phi(w) \in \mathcal{O}$ and $x' = \phi'(w) \in \mathcal{O}'$. Let $K_x = \mathrm{Stab}_{K_{\mathbb{C}}}(x)$ and $K'_{x'} = \mathrm{Stab}_{K'_{\mathbb{C}}}(x')$. Then*

(i) $\phi'^{-1}(x') = \phi'^{-1}(x') \cap W^\circ = \phi'^{-1}(x') \cap \phi^{-1}(\overline{\mathcal{O}}) = \phi'^{-1}(x') \cap \phi^{-1}(\mathcal{O})$ is a single $K_{\mathbb{C}}$-orbit where $K_{\mathbb{C}}$ acts freely.

(ii) There is a (surjective) group homomorphism

$$\alpha\colon K'_{x'} \to K_x \quad \text{such that} \quad k' \cdot w = \alpha(k'^{-1}) \cdot w \quad \forall k' \in K_x. \tag{35}$$

In particular,

$$\mathrm{Stab}_{K_{\mathbb{C}} \times K'_{\mathbb{C}}}(w) = K_x \times_\alpha K'_{x'} := \{ (\alpha(k'), k') \mid k' \in K'_{x'} \}. \tag{36}$$

Proof. By Theorem A.4(i) $\phi'^{-1}(x') = \phi'^{-1}(x') \cap W^\circ$ and Lemma A.3(ii) states that this is a single $K_{\mathbb{C}}$-orbit where $K_{\mathbb{C}}$ acts freely. This proves (i). Fix $k' \in K_{x'}$, $k' \cdot w \in \phi'^{-1}(x')$. By (i), there is a unique $k \in K_{\mathbb{C}}$ such that $k \cdot w = k' \cdot w$. Since $x = \phi(k' \cdot w) = \phi(k \cdot w) = k \cdot \phi(w) = k \cdot x$, we have $k \in K_x$. We define $\alpha(k') = k^{-1}$. It is straightforward to check that α is a group homomorphism and (36) holds.

Now we prove that α is surjective. Fix $k \in K_x$. Since $k \cdot w \in \phi^{-1}(x) \cap W^\circ$, there is an element $k' \in K'_{\mathbb{C}}$, such that $k' \cdot w = k \cdot w$ by Lemma A.3(iii). It is clear that that $k' \in K'_{x'}$ so $k \cdot w = \alpha(k')^{-1} \cdot w$. Note that $K_{\mathbb{C}}$-action on $\phi'^{-1}(x')$ is free, we have $\alpha(k')^{-1} = k$, i.e. α is surjective. This completes the proof of (ii) and the corollary. □

A.6. We discuss the scheme theoretical properties of (2).

Let $R = W - W^\circ$ be the set of matrices without full rank. Let $\mathcal{N} = \phi^{-1}(0) \cap W^\circ$ and $\partial\mathcal{N} := \overline{\mathcal{N}} - \mathcal{N}$. By Theorem A.4, $\mathcal{N}$ is a single $K'_{\mathbb{C}}$-orbit, $\overline{\mathcal{N}} = \phi^{-1}(0)$ and $\partial\mathcal{N} = R \cap \overline{\mathcal{N}}$. We state some well known geometry properties of the null fiber $\overline{\mathcal{N}}$.

Theorem A.7 ([5] [16] [24])**.** *Let (G, G') be a real reductive dual pair in stable range as in Table 1.*

(i) We have $\mathbb{C}[W] = \mathcal{H} \otimes \mathcal{S}(\mathfrak{p})$ where $\mathcal{H}$ is the space of $K'_{\mathbb{C}}$-harmonic. In particular, the map $\phi\colon W \to \mathfrak{p}^*$ is a faithfully flat morphism.

(ii) The scheme theoretical fiber $W \times_{\mathfrak{p}^*} \{ 0 \}$ is reduced[3], i.e. $\overline{\mathcal{N}} = W \times_{\mathfrak{p}^*} \{ 0 \}$.

(iii) If the dual pair is not (††) in Section 1.6, then $\overline{\mathcal{N}}$ is normal and $\partial\mathcal{N}$ has codimension at least 2 in $\overline{\mathcal{N}}$. □

[3]Our base field is $\mathbb{C}$ so geometrically reduced (resp. geometrically normal) is equivalent to reduced (resp. normal).

Let $\mathcal{O}$ be a nilpotent $K_{\mathbb{C}}$-orbit in $\mathfrak{p}^*$. Let $Z := W \times_{\mathfrak{p}^*} \overline{\mathcal{O}}$ (resp. $Y = W \times_{\mathfrak{p}^*} \mathcal{O}$) be the scheme theoretic inverse image of $\overline{\mathcal{O}}$ (resp. $\mathcal{O}$).

Lemma A.8. *(i) The scheme Z and Y are reduced.*
(ii) Suppose the dual pair is not (††). Then Y is normal. If $\overline{\mathcal{O}}$ is normal, then Z is normal.

By the above lemma, we can also view $Z = \phi^{-1}(\overline{\mathcal{O}})$ and $Y = \phi^{-1}(\mathcal{O})$ as the set theoretical fibers.

Proof. (i) Since $\phi\colon W \to \mathfrak{p}^*$ is faithfully flat, $\phi|_Z\colon W \times_{\mathfrak{p}'^*} \overline{\mathcal{O}} \to \overline{\mathcal{O}}$ is faithfully flat.

Let E_r (resp. E_n) be the set of elements in W which is geometrically reduced (resp. geometrically normal) in the fiber of $\phi(w)$, i.e.

$$E_r := \{\, w \in W \mid w \text{ is reduced in } W \times_{\mathfrak{p}^*} \phi(w) \,\}.$$

By Theorem 12.1.6 in [6], E_r (resp. E_n) is open in W. By Theorem A.7(ii) and (iii), $\overline{\mathcal{N}} \subset E_r$ (resp. $\overline{\mathcal{N}} \subset E_n$).

We claim that $Z \subset E_r$ and $Z \subset E_n$. We only prove $Z \subset E_r$. The proof of $Z \subset E_n$ is the same. Clearly E_r is $K_{\mathbb{C}} \times K'_{\mathbb{C}}$-invariant. It suffices to prove that E_r contains every closed point $z \in Z$. In order to show that $z \in E_r$, it suffices to show that $E_r \cap \overline{K_{\mathbb{C}} K'_{\mathbb{C}} \cdot z} \neq \emptyset$. Since $\phi\colon W \to \mathfrak{p}^*$ is an affine quotient map, it maps $K_{\mathbb{C}}$-invariant closed subset in W to closed subset in $\mathfrak{p}^*$ (see [27, Corollary 4.6]). Therefore

$$\phi(\overline{K_{\mathbb{C}} K'_{\mathbb{C}} \cdot z}) = \overline{\phi(K_{\mathbb{C}} K'_{\mathbb{C}} \cdot z)} = \overline{K_{\mathbb{C}} \phi(z)} \ni 0.$$

Hence $\emptyset \neq \overline{K_{\mathbb{C}} K'_{\mathbb{C}} \cdot z} \cap \overline{\mathcal{N}} \subset \overline{K_{\mathbb{C}} K'_{\mathbb{C}} \cdot z} \cap E_r$. This proves our claim.

We recall Proposition 11.3.13 in [6].

Proposition. *Suppose $f\colon X \to Y$ is a finitely presented flat morphism, and $y = f(x)$. Then x is reduced (resp. normal) in X if*
(i) y is reduced (resp. normal) in Y and
(ii) x is reduced (resp. normal) in $X \times_Y \{\, y \,\}$.

We continue with the proof of Lemma A.8. Since $\overline{\mathcal{O}}$ is reduced, $Z \subset E_r$ and the above proposition proves (i). Since $\mathcal{O}$ is smooth, it is normal (resp. $\overline{\mathcal{O}}$ is normal by assumption). $Y \subset Z \subset E_n$ proves (ii). This completes the proof of Lemma A.8. □

We state a consequence of Lemma A.8.

Lemma A.9. *$\overline{\mathcal{O}'} = Z/K_{\mathbb{C}}$, or equivalently, $\mathbb{C}[\overline{\mathcal{O}'}] = (\mathbb{C}[W] \otimes_{S(\mathfrak{p})} \mathbb{C}[\overline{\mathcal{O}}])^{K_{\mathbb{C}}}$.*

Proof. By Lemma A.8(i), $\mathbb{C}[Z]^{K_{\mathbb{C}}} = (\mathbb{C}[W] \otimes_{S(\mathfrak{p})} \mathbb{C}[\overline{\mathcal{O}}])^{K_{\mathbb{C}}}$ is reduced. The lemma follows from the fact that ϕ' is an affine quotient map onto its image [31]. Also see Proposition 3(2) in [26]. □

Let Z° be the open dense $K_{\mathbb{C}} \times K'_{\mathbb{C}}$-orbit in $Y = \phi^{-1}(\mathcal{O})$. Let $\partial Z^\circ = Y - Z^\circ$. By Theorem A.4(i), $\partial Z^\circ = R \cap Y$ where $R = W - W^\circ$ is the set of elements without full rank.

Lemma A.10. *We have* $\operatorname{codim}(Y, \partial Z^\circ) \geq 2$.

Proof. If $\partial Z^\circ = \emptyset$, then there is nothing to prove. Now suppose $\partial Z^\circ \neq \emptyset$. We reduce to consider irreducible components separately. Hence we assume all the schemes below are irreducible.

Let $\overline{\mathcal{N}} = \phi^{-1}(0)$ be the closed null cone and let $\partial\mathcal{N} = \overline{\mathcal{N}} - \mathcal{N}$. It is known that $\partial\mathcal{N} = R \cap \overline{\mathcal{N}}$ and $\operatorname{codim}(\overline{\mathcal{N}}, \partial\mathcal{N}) = \dim \overline{\mathcal{N}} - \dim \partial\mathcal{N} \geq 2$.

We recall that Y is open dense in Z and $\phi|_Z : Z \to \overline{\mathcal{O}}$ is faithfully flat. Therefore

$$\dim Y = \dim Z = \dim \overline{\mathcal{O}} + \dim \overline{\mathcal{N}}.$$

By our assumption $\partial Z^\circ \neq \emptyset$. The map $\phi|_{R\cap Z} \colon R \cap Z \to \overline{\mathcal{O}}$ is dominant since its image contains $\mathcal{O} = \phi(\partial Z^\circ)$. Moreover $\partial N = (\phi|_{R\cap Z})^{-1}(0)$. By the upper semi-continuity theorem of fiber dimension,

$$\dim \partial Z^\circ \leq \dim R \cap Z \leq \dim \overline{\mathcal{O}} + \dim \partial\mathcal{N}.$$

Therefore

$$\operatorname{codim}(Y, \partial Z^\circ) = \dim Y - \dim \partial Z^\circ \geq \dim \overline{\mathcal{N}} - \dim \partial N \geq 2.$$

This proves the lemma. □

Appendix B. Invariants of contragredient representations

In this section, we state well-known facts about the invariants of contragredient representations. Since the proofs are not easily available elsewhere, we supply them as well.

Let G be a real reductive group and let K be its maximal compact subgroup. Let (ρ, V) be $(\mathfrak{g}, K)$-module of finite length. Let (ρ^*, V^*) be its contragredient representation where $V^* = \mathrm{Hom}(V, \mathbb{C})_{K-\text{finite}}$.

We recall the variety $\mathrm{V}_\mathbb{C}(V)$ associated to the annihilator ideal $\mathrm{Ann}\, V = \mathrm{Ann}_{\mathcal{U}(\mathfrak{g})} V$. It is a subvariety in the nilpotent cone of $\mathfrak{g}^*$ cut out by the graded ideal $\mathrm{Gr}\,(\mathrm{Ann}\, V)$.

Proposition B.1. *We have* $\mathrm{V}_\mathbb{C}(V^*) = \mathrm{V}_\mathbb{C}(V)$.

Proof. Let ι be the anti-involution ι on $\mathcal{U}(\mathfrak{g})$ such that $\iota(X) = -X$ and $\iota(XY) = YX$ for all $X, Y \in \mathfrak{g}$. Passing to the graded module $\mathbb{C}[\mathfrak{g}^*] = S(\mathfrak{g}) = \mathrm{Gr}\,\mathcal{U}(\mathfrak{g})$, $\mathrm{Gr}\,\iota$ is given by pre-composing the map on $\mathfrak{g}^*$ defined by $\mathfrak{g}^* \ni \lambda \mapsto -\lambda$. Then $\iota(\mathrm{Ann}\, V) = \mathrm{Ann}\, V^*$ and $\mathrm{V}_\mathbb{C}(V^*) = -\mathrm{V}_\mathbb{C}(V)$. On the other hand, $\mathrm{V}_\mathbb{C}(V)$ is a union of nilpotent $G_\mathbb{C}$-orbits so $\mathrm{V}_\mathbb{C}(V) = -\mathrm{V}_\mathbb{C}(V)$. This proves the proposition. □

B.2. In order to define the filtration on (ρ^*, V^*) we need to review [1]. There is a Chevalley involution $C \in \mathrm{Aut}(G)$ such that $C(K) = K$, $\mathrm{Ad}_C(\mathfrak{k}) = \mathfrak{k}$ and $\mathrm{Ad}_C(\mathfrak{p}) = \mathfrak{p}$. Furthermore for every semi-simple element $g \in G$, $C(g)$ is conjugate to g^{-1} in G. We define a representation (ρ^C, V^C) where $V^C = V$ and $\rho^C(k) = \rho(C(k))$ for all $k \in K$ and $\rho^C(X) = \rho(\mathrm{Ad}_C(X))$ for all $C \in \mathcal{U}(\mathfrak{g})$. We will call (ρ^C, V^C) the representation of (ρ, V) *twisted* by C. Corollary 1.2 in [1] states that (ρ^C, V^C) is isomorphic to the dual representation (ρ^*, V^*).

Clearly, if $\mathcal{O}$ is a nilpotent $K_\mathbb{C}$-orbit in $\mathfrak{p}^*$ generated by x, then $\mathrm{Ad}_C^*(\mathcal{O})$ is a nilpotent $K_\mathbb{C}$-orbit in $\mathfrak{p}^*$ generated by $\mathrm{Ad}_C^*(x)$. Moreover, $K_{\mathrm{Ad}_C^*(x)} = C(K_x)$. For every K_x-module (resp. K_x-character) χ_x, $\chi_x \circ C$ is a $K_{\mathrm{Ad}_C^*(x)}$-module (resp. $K_{\mathrm{Ad}_C^*(x)}$-character).

Proposition B.3. *Let $C \in \mathrm{Aut}(G)$ be the Chevalley involution. We have*

(i) $\mathrm{AV}(\rho^*) = \mathrm{Ad}_C^*(\mathrm{AV}(\rho))$,
(ii) $\mathrm{AC}(\rho^*) = \mathrm{Ad}_C^*(\mathrm{AC}(\rho))$,
(iii) Suppose $x \in \mathfrak{p}^*$ generates an open orbit in $\mathrm{AV}(\rho)$. Let χ_x be the isotropy character of K_x at x. Then $\chi_x \circ C$ is the isotropy character at $\mathrm{Ad}_C^*(x)$.

Proof. Let V_j be an good filtration of (ρ, V). Then V_j is also a good filtration of (ρ^C, V^C) since $C(K) = K$ and $\mathrm{Ad}_C(\mathfrak{g}) = \mathfrak{g}$. Therefore, the $(\mathcal{S}(\mathfrak{p}), K_\mathbb{C})$ actions on $\mathrm{Gr}\, V^C$ is given by precomposing C, i.e. $\mathrm{Gr}\, V^C = \mathrm{Gr}\, V \circ C$. This proves the lemma. □

B.4. Let (ρ, V) be an irreducible $(\mathfrak{g}, \widetilde{K})$-module which is a quotient of $\mathscr{Y}$. Let $V_j = \mathcal{U}_j(\mathfrak{g})V_\tau$ be the filtration generated by lowest degree $\widetilde{K}$-type V_τ. By the definition of C, $\tau \circ C|_{\widetilde{K}} = \tau^*$. We fix a $(\mathfrak{g}, \widetilde{K})$-module isomorphism between V^C and V^*. Since V_τ has multiplicity 1 in V, V_{τ^*} has multiplicity 1 in V^* too. Therefore the filtration $V_j^C := V_j$ defined on V^C is same as the filtration $V_j^* = \mathcal{U}_j(\mathfrak{g})V_{\tau^*}$ defined on V^*.

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